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Thesis

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Theme

Étude de quelques problèmes inverses associés à certains problèmes aux limites

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M. Farid

ملخص: معادلات الانتشار الكسري التي تُعمّم معادلات الانتشار الكلاسيكية باستبدال المشتقة الزمنية من الدرجة الأولى بمشتق كسري من الدرجة $0<\alpha\leq 1$ تُوفّر إطارًا فعّالًا لنمذجة الظواهر دون الانتشارية في تطبيقات مُتنوّعة، بدءًا من النقل الكيميائي والمواد اللزجة المرنة، وصولًا إلى الأنظمة البيولوجية والهندسة النووية. إلا أن نواتها غير المحلية والمفردة بشكل ضعيف تُعيق الاستخدام المُباشر للعديد من التقنيات التحليلية القياسية. وعلى وجه الخصوص، تُشكّل المسائل العكسية، والخاصة في استعادة حدود أو مُعاملات مصدرية غير معروفة من قياسات متكاملة أو نقطية، تحديات إضافية، إذ يجب استيعاب المشتقات الكسرية، وغالبًا ما تكون مسألة القيم الحدودية غير مُترافقة ذاتيًا.

ترتكز هذه الأطروحة على مشكلتين عكسيتين رئيسيتين لمعادلة الانتشار الكسري الزمني أحادية البعد:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)=u_{xx}(x,t)+F(x,t,u(x,t)),\quad 0<x<1,\ 0<t\leq T,$$

تخضعان لعائلة غير محلية من الشروط الحدودية وشرط إضافي تكاملي، حيث يمثل $\frac{\partial^\alpha}{\partial t^\alpha}$ المشتق الكسري بالنسبة للزمن t. في المسألة العكسية الأولى، يأخذ المصدر للشكل المنفصل التالي $F(x,t,u(x,t))=r(t)f(x,t)$ ويتم البحث عن الثنائية $\{u(,t),r(t)\}$ ، أما في المسألة العكسية الثانية، فإن دالة المصدر تعطى بـ $F(x,t,u(x,t))=-p(t)u(,t)+S(x,t)$ والهدف هو البحث عن الثنائية $\{u(x,t),p(t)\}$. دُرست كلتا المسألتين بافتراضات الحد الأدنى من الانتظام والتوافق للمعطيات. بعد مراجعة النظرية الطيفية ذات الصلة بمؤثر ستورم-ليوفيل ذو الشروط الحدودية المنتظمة ولكن غير المنتظمة بشدة، استخدمنا طريقة فوربيه المُعممة، ووسّعنا الحلول في الأنظمة ثنائية التعامد للدوال الجذرية ونظرية النقطة الثابتة بناءً على تحديدات لدالة ميتاغ-ليفلر لإثبات الوجود والوحدانية والارتباط المستمر للحل بالنسبة لمعطيات المسألة. استُكملت النتائج النظرية بمخططات الفروق المحدودة وخوارزمية طيفية مبنية على كثيرات حدود ليجندر المُزاحة، مع أمثلة عددية تُؤكد دقة واستقرار الطرق المقترحة.

كلمات مفتاحية: معادلات الانتشار الكسري، شروط الحدود غير المحلية، المشتق الكسري كبوتو المعمم، المشتق الكسري المطابق، طريقة فوربيه المعممة، نظرية النقطة الثابتة لباناخ، مخطط الفروق المحدودة، طريقة تجميع ليجندر، طريقة أولر العكسية، مشكلة الطيف غير المترافق ذاتيًا، الأنظمة الثنائية المتعامدة.

Time-fractional diffusion equations (TFDEs), which generalize classical diffusion equations by replacing the first-order time derivative with a fractional-order operator of order $0 < \alpha \leq 1$, provide an effective framework for modeling subdiffusive phenomena in diverse applications-from chemical transport and viscoelastic materials to biological systems and nuclear engineering. Their nonlocal, weakly singular kernels, however, obstruct the direct use of many standard analytical techniques. In particular, inverse problems-recovering unknown source terms or coefficients from integral or pointwise measurements-pose additional challenges, as one must accommodate fractional-order derivatives and often non-self-adjoint boundary value problem.

This thesis focuses on two main inverse problems for the one-dimensional time-fractional diffusion equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}}(x,t) = u_{xx}(x,t) + F(x,t,u), \quad 0 < x < 1, \; 0 < t \leq T,$$

subject to nonlocal family of boundary conditions and an integral over-determination, where $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ represents a fractional derivative. In the first, the source takes the separable form $F(x,t,u) = r(t) f(x,t)$, and one seeks the pair $\{u(x,t), r(t)\}$. In the second, $F(x,t,u) = -p(t) u(x,t) + S(x,t)$ and the goal is to recover the time-dependent coefficient $p(t)$ alongside u. Both problems are studied under minimal regularity and compatibility assumptions on the data. After reviewing relevant spectral theory for Sturm-Liouville operators with regular but not strongly regular boundary conditions, we employ a generalized Fourier method, expanding solutions in biorthogonal systems of root functions and fixed-point arguments based on Mittag-Leffler estimates to establish existence, uniqueness, and continuous dependence. Theoretical results are complemented by finite-difference schemes and a spectral algorithm based on shifted Legendre polynomials, with numerical examples confirming accuracy and stability.

Keywords: Fractional diffusion equations, nonlocal boundary conditions, Generalized Caputo fractional derivative, Conformable fractional derivative, Generalized Fourier method, Banach fixed-point theorem, Finite-difference scheme, Legendre collocation method, Backward Euler method, non-self-adjoint spectral problem, biorthogonal systems.

Les équations de diffusion fractionnaires temporelles (TFDE), qui généralisent les équations de diffusion classiques en remplaçant la dérivée temporelle du premier ordre par un opérateur d'ordre fractionnaire d'ordre $0 < \alpha \leq 1$, offrent un cadre efficace pour la modélisation des phénomènes subdiffusifs dans diverses applications, du transport chimique et des matériaux viscoélastiques aux systèmes biologiques et au génie nucléaire. Leurs noyaux non locaux et faiblement singuliers entravent cependant l'utilisation directe de nombreuses techniques analytiques standard. En particulier, les problèmes inverses (récupération de termes sources ou de coefficients inconnus à partir de mesures intégrales ou ponctuelles) posent des défis supplémentaires, car il faut prendre en compte les dérivées d'ordre fractionnaire et souvent les problèmes aux limites non auto-adjoints.

Cette thèse se concentre sur deux principaux problèmes inverses pour l'équation de diffusion fractionnaire en temps unidimensionnelle

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = u_{xx}(x,t) + F(x,t,u), \quad 0 < x < 1, \ 0 < t \leq T,$$

soumis à une famille non locale de conditions aux limites et à une surdétermination intégrale, où $\frac{\partial^\alpha}{\partial t^\alpha}$ représente une dérivée fractionnaire. Dans le premier cas, la source prend la forme séparable $F(x,t,u) = r(t) f(x,t)$, et on cherche le couple $\{u(x,t), r(t)\}$. Dans le second cas, $F(x,t,u) = -p(t) u(x,t) + S(x,t)$ et l'objectif est de retrouver le coefficient dépendant du temps $p(t)$ aux côtés de u. Les deux problèmes sont étudiés sous des hypothèses minimales de régularité et de compatibilité sur les données. Après avoir examiné la théorie spectrale pertinente pour les opérateurs de Sturm-Liouville avec des conditions aux limites régulières mais non fortement régulières, nous utilisons une méthode de Fourier généralisée, étendant les solutions aux systèmes biorthogonaux de fonctions racines et les arguments de point fixe basés sur les estimations de Mittag-Leffler pour établir l'existence, l'unicité et la dépendance continue. Les résultats théoriques sont complétés par des schémas aux différences finies et un algorithme spectral basé sur des polynômes de Legendre décalés, avec des exemples numériques confirmant la précision et la stabilité.

Mots-clés: Équations de diffusion fractionnaire, Conditions aux limites non locales, Dérivée fractionnaire de Caputo généralisée, Dérivée fractionnaire conformable, Méthode de Fourier généralisée, Théorème du point fixe de Banach, Schéma aux différences finies, Méthode de collocation de Legendre, Méthode d'Euler rétrograde, Problème spectral non auto-adjoint, Systèmes biorthogonaux.

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General introduction

Parabolic partial differential equations with fractional time derivatives of order less than 1, known as time-fractional diffusion equations (TFDEs), have become a crucial tool for modeling slow diffusion (subdiffusion) processes in various fields, including chemistry, physics, viscoelasticity, biology, and nuclear power engineering, [8, 38, 39]. The challenges associated with TFDEs are more pronounced because many standard methods cannot be directly applied to non-classical derivatives. The difficulty arises from the definition of fractional-order derivatives, which fundamentally involve an integral with a weakly singular kernel.

It is important to highlight that the first theoretical results for the inverse problem of determining coefficients in TFDEs were established in [12, 28, 34, 47, 50]. Inverse source problems for TFDEs have been extensively studied under various initial, boundary, and over-determination conditions. The problem of identifying a space-dependent source term from the final temperature distribution has been investigated in [5, 7, 14, 15, 32, 49], while the recovery of a space-dependent source term from total energy measurements has been discussed in [10, 36, 37].

For inverse problems involving TFDEs, the identification of a time-dependent source term from temperature measurements at a selected spatial point has been considered in [6, 25], while the determination of a time-dependent source term using an integral-type over-determination condition has been explored in [3, 4, 18, 21]. Additionally, the identification of initial and boundary data from final measurements in the initial boundary value problem for the time-fractional heat equation has been examined in [2, 33] and [11], respectively.

In the study of inverse source problems for TFDEs, several approaches have been explored in the literature, one of which is the generalized Fourier method. The works [2–5, 21, 34] investigate inverse source problems from the perspective of spectral analysis, where the temperature distribution is expanded in terms of root functions (eigenfunctions and associated functions) of a spectral problem with boundary conditions relevant to the given problem. When the boundary conditions involve nonlocal characteristics, classical self-adjoint eigenfunction expansion results cannot be directly applied to the auxiliary spectral problem, necessitating further investigation into eigenfunction expansions [41]. Non-self-adjoint operators commonly arise in the modeling of dissipative processes [19, 48]. In many cases, nonlocal conditions provide a more realistic framework for addressing physical problems than traditional local conditions, further motivating the study of nonlocal boundary-value problems.

In this thesis, we are interested with the one-dimensional time-fractional diffusion equation

$$ {}^{c}\mathcal{D}_{t}^{\alpha,\rho}u\left(x,t\right)=u_{xx}\left(x,t\right)+F\left(x,t,u\right),\ \ \left(x,t\right)\in\Omega_{T}, \tag{1} $$

subject to the initial condition

$$ u\left(x,0\right)=\varphi\left(x\right),\ \ 0<x<1, \tag{2} $$

and the family of nonlocal boundary conditions

$$ \begin{cases} u\left(0,t\right)=u\left(1,t\right), \\ \beta u_{x}\left(0,t\right)=u_{x}\left(1,t\right), \end{cases} \quad 0<t\leq T, \tag{3} $$

where $-1<\beta<1$ and $\Omega_T:=\{(x,t):0<x<1,\ 0<t\leq T\}$ for some fixed $T>0$, ${}^{c}\mathcal{D}_{t}^{\alpha,\rho}$ stands for left-sided generalized Caputo fractional derivative of order $0<\alpha\leq 1$ and $\rho>0$ is a real constant, $F\left(x,t,u\right)$ is the source term and $\varphi\left(x\right)$ is the initial data. The fractional derivative in (1) is a generalization of left Caputo and Caputo-Hadamard fractional derivatives, which can be obtained by taking $\rho=1$ and $\rho\to 0^{+}$, respectively. The nonlocal boundary conditions (3) are regular but not strongly regular. For $\beta=0$, the nonlocal boundary conditions (3) are well-known and called in literature as Samarskii-Ionkin conditions, [20]. More general boundary conditions of the type (3) have been considered in [17, 24, 41, 44].

Our goal in this thesis is to determine the source term F in two cases, together with $u\left(x,t\right)$ under the additional integral measurement:

$$ \int_{0}^{1}u\left(x,t\right)dx=g\left(t\right),\ t\in\left[0,T\right], \tag{4} $$

where g belongs to $AC\left[0,T\right]$, the space of absolutely continuous functions. The integral condition (4) naturally arises and serves as supplementary information for identifying the source term. This type of condition is applicable in various physical contexts, including chemical engineering, thermo-elasticity, heat conduction and diffusion processes, and fluid flow in porous media [21].

When the function F is given, the problem of finding $u(x,t)$ from the initial boundary value problem given by (1)-(3) is called the direct problem . The inverse problem given by (1)-(4) is formulated when the function F is unknown. The following two inverse problems will be studied.

Inverse time-dependent source problem:

If we take the unknown function F to be $F(x,t,u) = r(t) f(x,t)$, the inverse time-dependent source problem is formulated as the problem of finding the pair $\{u(x,t), r(t)\}$ satisfying (2)-(4) and

$$ {}^{c}\mathcal{D}_t^{\alpha,\rho} u(x,t) = u_{xx}(x,t) + r(t) f(x,t), \quad (x,t) \in \Omega_T, \tag{5}$$

where $f(x,t)$ is a given function and $r(t)$ is an unknown function.

The study of inverse source problems with same conditions has been considered earlier [22,46]. In [22], the inverse source problem (2)-(4) and (5) was studied using the time-fractional derivative in the Riemann-Liouville sense. In [46], the same inverse source problem is studied for $\alpha = \rho = 1$. The case $0 < \alpha < 1$ and $\rho \neq 1$ is considered in our article [40] for the first time.

Inverse time-dependent coefficient problem:

If we take the unknown function F to be $F(x,t,u) = -p(t) u(x,t) + S(x,t)$, the inverse problem is formulated as the problem of finding the pair $\{u(x,t), p(t)\}$ satisfying (2), (4) and

$$\mathcal{D}_t^{(\alpha)} u(x,t) = u_{xx}(x,t) - p(t) u(x,t) + S(x,t), \quad (x,t) \in \Omega_T, \tag{6}$$

where $\mathcal{D}_t^{(\alpha)}$ represent the left-conformable fractional derivative of order $0 < \alpha \leq 1$ with respect to t, $S(x,t)$ is a given function and $a(t)$ is an unknown function.

The rest of our thesis is organized as follows: in Chapter 1, we provide some preliminaries and basic result needed for the forthcoming chapters.

In Chapter 2, we study a spectral problem for Sturm-Liouville operator with two-point boundary conditions. We introduce some necessary properties regarding: regular boundary conditions and biorthogonal systems in a Hilbert space. We make a detailed study of the completeness property and the fundamental property of the system of root functions. Finally, we introduce the non-self-adjoint spectral problem that we use in Chapter 3 and which is a key element of a paper published in an international journal [40].

In Chapter 3, we study the time-dependent source inverse problem (2)- (4) and (5). The peculiarity of this inverse problem is that the system of eigenfunctions is not complete, but the system of eigenfunctions and associated functions forms a basis in $L^2(0,1)$. Under certain natural conditions of regularity and consistency of the input data, the existence, uniqueness, and continuous data dependence of the solution are shown using the generalized Fourier method, Mittag-Leffler function estimates, and the Banach contraction principle. This chapter is a draft of an article titled "*An inverse time-dependent source problem for a time-fractional diffusion equation witn nonlocal boundary conditions*" published in an international journal [40].

In Chapter 4, we study an inverse problem of determining the time-dependent coefficient in one-dimensional time-fractional reaction-diffusion equation with nonlocal boundary and overdeterminarion conditions. The time-fractional derivative is described in the conformable sense. Under some assumptions on the input data, the well-posedness of this inverse time-dependent coefficient problem is shown by using Fourier's method and Banach's contraction mapping principle.

In Chapter 5, we discus the finite difference approximation for the inverse time-dependent source problem (2)- (4) and (5).

In the last chapter, an efficient algorithm is proposed for solving the inverse time-dependent source problem (2)- (4) and (5). This algorithm is based on shifted Legendre polynomials of the first kind. This inverse problem is reduced to a linear system of first order differential equations and the Backward Euler method is applied to solve this system. Finally, some numerical examples are presented to confirm the reliability and effectiveness of this algorithm.

We conclude this thesis with a general conclusion and some perspectives.

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce the mathematical tools essential for a thorough understanding of the thesis. We begin with a review of key concepts from functional analysis and Fourier analysis, followed by an introduction to fundamental definitions and basic properties related to fractional calculus and inverse problems.

1.1 Basic results of Banach spaces

Definition 1.1. Let E be a vector space over $\mathbb{R}$. A real-valued function $\|\cdot\|$ defined on E and satisfying the following conditions is called a norm:

(1) $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in E$ and $\alpha \in \mathbb{R}$.

(3) $\|x + y\| \leq \|x\| + \|y\|, \ \forall x, y \in \mathbb{R}$.

$(E, \|\cdot\|)$, vector space E equipped with $\|\cdot\|$ is called a normed space.

Definition 1.2. A normed space E is called a Banach space, if its every Cauchy sequence is convergent.

1.1.1 Examples of Banach spaces

Example 1.1. Let $a, b \in \mathbb{R}$ with $a < b$.

1. The vector space $\mathcal{C}([a,b])$ of all real-valued continuous functions defined on $[a,b]$ is a Banach space with respect to the following norm:

$$\|f\|_{\mathcal{C}([a,b])} := \max_{a \leq x \leq b} |f(x)|.$$

2. The vector space $\mathcal{C}^n([a,b])$ of all real-valued n time continuously differentiable functions defined on $[a,b]$ is a Banach space with respect to the following norm:

$$\|f\|_{\mathcal{C}^n([a,b])} := \sum_{k=0}^{n} \left\|f^{(k)}\right\|_{\mathcal{C}([a,b])} = \sum_{k=0}^{n} \max_{a\leq x\leq b} \left|f^{(k)}(x)\right|_{\mathcal{C}([a,b])}.$$

Definition 1.3. Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we denote $L^p(a,b)$ the space of Lebesgue's integrable functions on $[a,b]$ such that:

$$L^p(a,b) := \left\{u : [a,b] \to \mathbb{R},\ u \text{ measurable and } \|u\|_{L^p(a,b)} < \infty\right\},$$

with

$$\|u\|_{L^p(a,b)} := \left(\int_a^b |u(x)|^p\, dx\right)^{1/p}.$$

The following theorem summarizes some properties of the L^p spaces:

Theorem 1.1. *Let $1 \leq p < +\infty$. Then*

1. *The space $L^p(a,b)$ endowed with the norm $\|\cdot\|_{L^p(a,b)}$ is a Banach space.*

2. *Holder's inequality: Let $1 \leq q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $u \in L^p(a,b)$ and $v \in L^q(a,b)$. Then, $uv \in L^1(a,b)$ and*

$$\int_a^b |u(x)\, v(x)|\, dx \leq \left(\int_a^b |u(x)|^p\, dx\right)^{1/p} \left(\int_a^b |v(x)|^q\, dx\right)^{1/q}.$$

Corollary 1.1. *The space $L^2(a,b)$ is a Hilbert space with the inner product*

$$\langle u, v\rangle_{L^2(a,b)} := \int_a^b u(x)\, v(x)\, dx, \quad \forall u, v \in L^2(a,b).$$

Moreover, the following Cauchy-Schwarz inequality holds:

$$\left|\langle u, v\rangle_{L^2(a,b)}\right| \leq \|u\|_{L^2(a,b)}\, \|v\|_{L^2(a,b)}, \ \forall u, v \in L^2(a,b),$$

where

$$\|u\|_{L^2(a,b)} := \left(\int_a^b |u(x)|^2 \, dx\right)^{1/2}.$$

Definition 1.4 ([31, 1.1.5]). Let $[a, b]$ be a finite interval. Then, $AC[a, b]$ is the space of absolute continuous functions on $[a, b]$, defined by

$$AC[a,b] = \left\{ f: [a,b] \to \mathbb{R} \text{ such that } f(x) = c + \int_a^x \varphi(t)\, dt, \quad \varphi \in L^1(a,b) \right\}.$$

1.2 Fixed point theorem

Definition 1.5 (Contraction Mapping). Let $(E, \|\cdot\|)$ be a Banach space. A mapping $F : E \to E$ is called a Lipschitz continuous mapping if there exists a number $L > 0$ such that

$$\|F(x) - F(y)\| \leq L \|x - y\|, \ \forall x, y \in E.$$

If $0 < L < 1$, then F is called a contraction mapping. L is called the contractivity coefficient of F .

Definition 1.6 (Fixed point). Let F be a mapping on a Banach space $(E, \|\cdot\|)$ into itself. $u \in E$ is called a fixed point if

$$F(u) = u.$$

Next, let us recall the well-known Fixed point contraction mapping theorem, for more see [57].

Theorem 1.2 (Fixed point contraction mapping theorem). *Let $(E, \|\cdot\|)$ be a Banach space and let F be a contraction mapping on E into itself with contractivity coefficient $0 < L < 1$. Then there exists only one point u in E such that $F(u) = u$, that F has a unique fixed point. Furthermore, for any $u \in E$ the sequence*

$$v, F(v), F^2(v), \ldots, F^k(v),$$

converges to the point u; that is

$$\lim_{k\to+\infty} F^{k}(v) = u.$$

1.3 Fractional calculus

This section provides a review of relevant definitions, notations in fractional calculus, and fundamental results for the reader's convenience.

Definition 1.7 ([29]). Let $[a,b]$ be a finite interval and $f\colon [a,b]\to\mathbb{R}$ be an integrable function. The generalized left fractional integral (in the sense of Katugampola) is defined by

$$\left(\mathcal{I}_a^{\alpha,\rho}\right) f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s)\, \frac{ds}{s^{1-\rho}}, \quad 0<\alpha<1, \quad \rho>0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function defined by $\Gamma(\alpha) := \int_0^{+\infty} t^{\alpha-1}e^{-t}\,dt$.

Definition 1.8 ([27]). Let $\rho>0$ and $f\in AC[a,b]$. The left generalized Caputo fractional derivative of f of order $0<\alpha<1$ is defined by

$${}_a^C\mathcal{D}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\alpha} f'(s)\, ds.$$

If $\alpha=\rho=1$, then ${}_a^c\mathcal{D}^{\alpha,\rho} f(t) = f'(t)$.

Theorem 1.3 ([27]). *Let $f\in AC[a,b]$, $0<\alpha<1$ and $\rho>0$. Then, we have:*

$$\mathcal{I}_a^{\alpha,\rho}\left({}_a^C\mathcal{D}^{\alpha,\rho} f(x)\right) = f(x) - f(a).$$

Definition 1.9 ([1]). Let $f\colon [0,+\infty[\to\mathbb{R}$ be a real valued function. The ρ-Laplace transform of f is defined by

$$\mathcal{L}_\rho\{f(t)\}(s) = \int_0^{+\infty} e^{-s\frac{t^\rho}{\rho}} f(t)\, \frac{dt}{t^{1-\rho}}, \quad \rho>0,$$

for all values of s, the integral is valid.

Theorem 1.4 ([1]). *If the ρ-Laplace transform of $f\colon [\,0,+\infty\,[\, \to \mathbb{R}$ exists for $s > c_1$ and the ρ-Laplace transform of $g\colon [\,0,+\infty\,[\, \to \mathbb{R}$ for $s > c_2$. Then, for any constants a and b, the ρ-Laplace transform of $af + bg$ exists and*

$$\mathcal{L}_\rho \left\{ af\left(t\right) + bg\left(t\right) \right\}\left(s\right) = a\mathcal{L}_\rho \left\{ f\left(t\right) \right\}\left(s\right) + b\mathcal{L}_\rho \left\{ g\left(t\right) \right\}\left(s\right), \quad \textit{for } s > \max\left\{c_1, c_2\right\}.$$

Definition 1.10 ([26]). Let f and g be two functions which are piecewise continuous at each interval $[0,T]$. We define the ρ-convolution of f and g by

$$\left(f * g\right)\left(t\right) = \int_0^t f\left[\left(t^\rho - s^\rho\right)^{1/\rho}\right] g\left(s\right) \frac{ds}{s^{1-\rho}}.$$

Theorem 1.5 ([26]). *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$. Then,*

$$\mathcal{L}_\rho \left\{ \left(f * g\right)\left(t\right) \right\} = \mathcal{L}_\rho \left\{ f\left(t\right) \right\} \mathcal{L}_\rho \left\{ g\left(t\right) \right\}.$$

Theorem 1.6 ([26]). *Let $\alpha > 0$ and $f \in AC\left[0,T\right]$. Then,*

$$\mathcal{L}_\rho \left\{ \left({}_0^C\mathcal{D}^{\alpha,\rho} f\right)\left(t\right) \right\}\left(s\right) = s^\alpha \mathcal{L}_\rho \left\{ f\left(t\right) \right\} - s^{\alpha-1} f\left(0\right).$$

Definition 1.11 ([16]). The Mittag-Leffler function of two parameters is defined as

$$E_{\xi,\eta}\left(x\right) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma\left(\xi k + \eta\right)}, \quad z \in \mathbb{C}, \quad Re\left(\xi\right) > 0, \quad Re\left(\eta\right) > 0.$$

For $\eta = 1$, the Mittag-Leffler function is reduced to classical one parameter Mittag-Leffler function, that is,

$$E_{\xi,1}\left(x\right) := E_\xi\left(x\right) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma\left(\xi k + 1\right)}.$$

Let $e_\xi\left(t,\mu\right) := E_\xi\left(-\mu t^\xi\right)$ and $e_{\xi,\eta}\left(t,\mu\right) := t^{\eta-1} E_{\xi,\eta}\left(-\mu t^\xi\right)$, where μ is a positive real number. The Mittag-Leffler functions $e_\xi\left(t,\mu\right)$, $e_{\xi,\eta}\left(t,\mu\right)$ for $0 < \xi \leq 1$, $0 < \xi \leq \eta \leq 1$, respectively, are completely monotone functions, i.e.

$$\left(-1\right)^n \frac{\partial^n}{\partial t^n}\left[e_\xi\left(t,\mu\right)\right] \geq 0 \text{ and } \left(-1\right)^n \frac{\partial^n}{\partial t^n}\left[e_{\xi,\eta}\left(t,\mu\right)\right] \geq 0, \quad n \in \mathbb{N}.$$

Using Theorem 1.6 in [43], we can have the following estimate

$$|\mu e_{\xi,\xi}(t,\mu)| \leq \frac{N\mu t^{\xi}}{t(1+\mu^{\xi})} \leq \frac{N}{t} \leq C, \quad t \in \left]\epsilon, T\right], \tag{1.1}$$

where $\epsilon > 0$, N and C are some constants.

Lemma 1.1 ([26]). *Let $\xi > 0$ and $\left|\frac{\lambda}{s^{\xi}}\right| < 1$. Then, we have:*

$$\mathcal{L}_{\rho}\left\{e_{\xi}\left(\frac{t^{\rho}}{\rho},\lambda\right)\right\} = \frac{s^{\alpha-1}}{s^{\alpha}+\lambda} \text{ and } \mathcal{L}_{\rho}\left\{e_{\xi,\xi}\left(\frac{t^{\rho}}{\rho},\lambda\right)\right\} = \frac{1}{s^{\alpha}+\lambda}.$$

Theorem 1.7 ([26]). *The Cauchy problem*

$$\begin{cases} {}_{0}^{C}\mathcal{D}^{\alpha,\rho}y(t) + \lambda y(t) = f(t), \quad t > 0, \quad 0 < \alpha < 1, \quad \rho > 0, \quad \lambda \in \mathbb{R}, \\ y(0) = y_0, \quad y_0 \in \mathbb{R}, \end{cases}$$

has the solution

$$y(t) = y_0 e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda\right) + \int_0^t e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda\right) f(s)\,\frac{ds}{s^{1-\rho}}.$$

1.4 Conformable fractional calculus

In this section, we start by recalling some concepts about conformable fractional calculus.

Definition 1.12 ([1]). Let $\varphi : [a,+\infty[\,\to \mathbb{R}$ is a given function and $\alpha \in \left]0,1\right]$. Then, the left-conformable fractional derivative of order α is defined by:

$$\mathcal{D}_t^{(\alpha)}(\varphi)(t) := \lim_{\varepsilon \to 0} \frac{\varphi\left(t + \varepsilon (t-a)^{1-\alpha}\right) - \varphi(t)}{\varepsilon}. \tag{1.2}$$

If $\mathcal{D}_t^{(\alpha)}(\varphi)(t)$ exists on $]a,+\infty[$, then $\mathcal{D}_t^{(\alpha)}(\varphi)(a) = \lim_{t\to a^+} \mathcal{D}_t^{(\alpha)}(\varphi)(t)$. If $a = 0$, the definition (1.2) is introduced by Khalil et al. [30]. In this case, we say that φ is α-differentiable.

Propriety 1.1 ([1,30]). For $f, g : [0, +\infty\, [\, \to \mathbb{R}$ and $0 < \alpha \leq 1$, we have the following properties:

$$\text{If } f \text{ is } \alpha\text{-differentiable, then } f \text{ is continuous.} \tag{1.3}$$

$$\mathcal{D}^{(\alpha)}(af + bg) = a\mathcal{D}^{(\alpha)}(f) + b\mathcal{D}^{(\alpha)}(g), \ a, b \in \mathbb{R}. \tag{1.4}$$

$$\mathcal{D}^{(\alpha)} t^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k-n)} t^{k-\alpha} & \text{If } k \in \mathbb{N} \text{ and } k > \alpha, \\ 0 & \text{If } k \in \mathbb{N} \text{ and } k < \alpha, \end{cases}, \tag{1.5}$$

where $\Gamma(\cdot)$ is the Euler Gamma function and $n < \alpha \leq n+1$.

$$\mathcal{D}^{(\alpha)}(C) = 0 \text{ where } C \text{ is a constant.} \tag{1.6}$$

$$\mathcal{D}^{(\alpha)}(fg) = f\mathcal{D}^{(\alpha)}(g) + g\mathcal{D}^{(\alpha)}(f). \tag{1.7}$$

$$\mathcal{D}^{(\alpha)}\left(\frac{f}{g}\right) = \frac{g\mathcal{D}^{(\alpha)}(f) - f\mathcal{D}^{(\alpha)}(g)}{g^2} \text{ with } g \neq 0. \tag{1.8}$$

If f is n times differentiable on $[a, +\infty\, [$ then we have:

$$\mathcal{D}^{(\alpha)}(f)(t) = (t-a)^{n+1-\alpha} f^{(n+1)}(t), \ n < \alpha \leq n+1. \tag{1.9}$$

Let $h(t) = (f \circ g)(t)$ such that f and g are α-differentiable functions, then

$$\mathcal{D}^{(\alpha)}(h)(t) = \mathcal{D}^{(\alpha)}(f)(g(t)) \cdot \mathcal{D}^{(\alpha)}(g)(t) \cdot g^{\alpha-1}(t). \tag{1.10}$$

Definition 1.13 ([30]). Let $\alpha \in\,]\,0, 1\,]$ and $\varphi : [0, +\infty\, [\, \to \mathbb{R}$ be real valued function. The left-conformable fractional integral of φ of order α from zero to t is defined by:

$$\mathcal{I}_\alpha \varphi(t) := \int_0^t s^{\alpha-1} \varphi(s)\, ds, \ \ t \geq 0, \tag{1.11}$$

Lemma 1.2 ([30]). *Let $\varphi : [0, +\infty\, [\, \to \mathbb{R}$ is a given function and $0 < \alpha \leq 1$. Then, for all $t > 0$, we have:*

1. *If φ is continuous, then $\mathcal{D}_t^{(\alpha)} [\mathcal{I}_\alpha \varphi(t)] = \varphi(t)$.*
2. *If φ is α-differentiable, then $\mathcal{I}_\alpha \left[\mathcal{D}_t^{(\alpha)}(\varphi)(t)\right] = \varphi(t) - \varphi(0)$.*

We introduce the following theorem, which is used further in this thesis.

Theorem 1.8. *Let $g : [0, +\infty\, [\, \to \mathbb{R}$ is a continuous function and $\gamma \in \mathbb{R}$. For all $0 < \alpha \leq 1$, the Cauchy problem:*

$$\begin{cases} \mathcal{D}_t^{(\alpha)} y(t) + \gamma y(t) = g(t), \\ y(0) = y_0. \end{cases} \tag{1.12}$$

admits a unique solution given by

$$y(t) = y_0 \exp\left(-\gamma \frac{t^\alpha}{\alpha}\right) + \int_0^t \exp\left(\gamma \frac{s^\alpha - t^\alpha}{\alpha}\right) s^{\alpha-1} g(s)\, ds. \tag{1.13}$$

1.5 Legendre polynomials

Legendre polynomials, also known as Legendre functions, are a class of orthogonal polynomials that constitute a special case of both ultra-spherical functions and Jacobi polynomials. These functions play a central role in various physical and mathematical problems, particularly those formulated in spherical coordinates. In such contexts, Legendre polynomials are essential for handling the angular components of functions, especially through their appearance in spherical harmonics, which are expressed in terms of these polynomials. There are two main types of Legendre polynomials.

In this thesis, we consider the first type of Legendre polynomials which is a solution of the following differential equation, see [13]:

$$\left(1 - x^2\right) y''(x) - 2xy'(x) + n(n+1)\, y(x) = 0. \tag{1.14}$$

Legendre polynomials of the first kind are denoted by $P_n(x)$.

Definition 1.14 ([13]). The Legendre polynomial of the first kind is a polynomial of degree n in x defined by the Rodriguez formula:

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} \left[\left(x^2 - 1\right)^n\right], \tag{1.15}$$

where x is a real or complex variable. Figure 1.1 shows the graphs of the first Legendre polynomials $P_n(x)$.

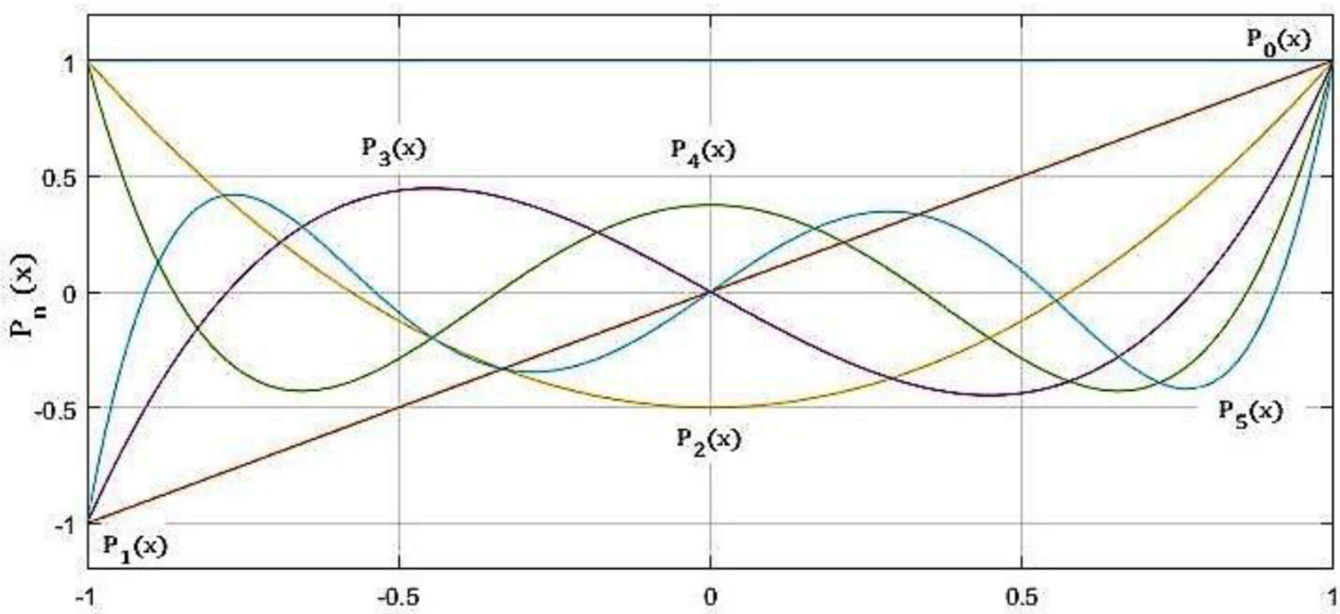

Figure 1.1: The Legendre polynomials $P_n(x)$ at different values of n.

Proposition 1.1 ([13]). *The analytic form of the Legendre polynomial of the first kind is given by:*

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2n-2k)!}{k!\,(n-k)!\,(n-2k)!} x^{n-2k}, \ n \in \mathbb{N}, \tag{1.16}$$

where $\left[\frac{n}{2}\right]$ *is the integer part of* $n/2$.

Proof. Using Newton's binomial law and Rodriguez's formula (1.14), we obtain:

$$\begin{aligned} P_n(x) &= \frac{1}{n!2^n} \frac{d^n}{dx^n} \left[\sum_{k=0}^{n} \frac{(-1)^k n! x^{2n-2k}}{k!\,(n-k)!} \right] \\ &= \frac{1}{2^n} \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!} \frac{d^n x^{2n-2k}}{dx^n} \\ &= \frac{1}{2^n} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!\,(n-k)!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}, \end{aligned}$$

Since $2n-2k$ in the combination must be greater than or equal to n, so $0 \leq k \leq \left[\frac{n}{2}\right]$. □

Definition 1.15 ([13]). The generating function of the Legendre polynomial $P_n(x)$ is given by:

$$\sum_{n=0}^{+\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}. \tag{1.17}$$

Proposition 1.2 ([13]). *The Legendre polynomials of the first kind $P_n(x)$ satisfy the following recurrence formula:*

$$\begin{cases} P_0(x) = 1, \\ P_1(x) = x \\ P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \ n \in \mathbb{N}. \end{cases} \tag{1.18}$$

Proof. By differentiating with respect to t the generating function (1.17), we obtain:

$$n \sum_{n=1}^{+\infty} t^{n-1} P_n(x) = \frac{x-t}{(1-2xt+t^2)\sqrt{1-2xt+t^2}}.$$

Multiplying both sides by $1-2xt+t^2$ and according to (1.17), we get:

$$n\left(1-2xt+t^2\right) \sum_{n=1}^{+\infty} t^{n-1} P_n(x) = \frac{x-t}{\sqrt{1-2xt+t^2}} = (x-t) \sum_{n=0}^{+\infty} t^n P_n(x).$$

Let each t be raised to the power n:

$$(n+1)\sum_{n=0}^{+\infty} t^n P_{n+1}(x) - 2nx \sum_{n=1}^{+\infty} t^n P_n(x) + (n-1) \sum_{n=2}^{+\infty} t^n P_{n-1}(x) = x \sum_{n=0}^{+\infty} t^n P_n(x) - \sum_{n=1}^{+\infty} t^n P_{n-1}(x).$$

By equating the coefficient of t^n, we obtain the recurrence formula (1.18). □

Lemma 1.3 ([13]). *The Legendre polynomials of the first kind form an orthogonal set on the interval $[-1, 1]$, such that*

$$\langle P_n, P_m \rangle = \int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{2}{2n+1} & \text{if } n = m. \end{cases} \tag{1.19}$$

Proof. 1. For $n \neq m$, P_n and P_m are solutions of the Legendre equation (1.11), then

$$\frac{d}{dx}\left[\left(1-x^2\right) P_n'(x)\right] + n(n+1) P_n(x) = 0, \tag{1.20}$$

$$\frac{d}{dx}\left[\left(1-x^2\right) P_m'(x)\right] + m(m+1) P_m(x) = 0. \tag{1.21}$$

Multiplying (1.20) by $P_m(x)$ and (1.21) by $P_n(x)$ and with the difference, we obtain:

$$\begin{aligned} &P_m(x)\left(\frac{d}{dx}\left[\left(1-x^2\right)P_n'(x)\right]+n(n+1)P_n(x)\right)\\ &-P_n(x)\left(\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]+m(m+1)P_m(x)\right)=0. \end{aligned} \tag{1.22}$$

Integrating the equation (1.22) with respect to x between -1 and 1, we obtain:

$$\begin{aligned} &\int_{-1}^{1}P_m(x)\left(\frac{d}{dx}\left[\left(1-x^2\right)P_n'(x)\right]\right)dx-\int_{-1}^{1}P_n(x)\left(\frac{d}{dx}\left[\left(1-x^2\right)P_m'(x)\right]\right)dx\\ &\qquad+\left[n(n+1)-m(m+1)\right]\int_{-1}^{1}P_n(x)P_m(x)\,dx=0. \end{aligned}$$

The proof is done using integration by parts.

2. For $n=m$ and using the generating function (1.17), we obtain:

$$\sum_{n=0}^{+\infty}t^nP_n(x)=\frac{1}{\sqrt{1-2xt+t^2}}, \tag{1.23}$$

$$\sum_{m=0}^{+\infty}t^mP_m(x)=\frac{1}{\sqrt{1-2xt+t^2}}. \tag{1.24}$$

Multiplying (1.23) by (1.24), we obtain:

$$\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}t^{n+m}P_n(x)P_m(x)=\frac{1}{1-2xt+t^2}.$$

Integrating both sides with respect to x from -1 to 1, we get:

$$\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}t^{n+m}\left(\int_{-1}^{1}P_n(x)P_m(x)\,dx\right)=\int_{-1}^{1}\frac{dx}{1-2xt+t^2}.$$

Since $\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0$ for $n \neq m$, then we have:

$$\begin{aligned}
\sum_{n=0}^{+\infty} t^{2n} \left(\int_{-1}^{1} P_n^2(x)\, dx \right) &= \int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} \\
&= -\frac{1}{2t} \left[\ln\left(1 - 2xt + t^2\right) \right]_{-1}^{1} \\
&= \frac{1}{t} \left[\ln(1+t) - \ln(1-t) \right] \\
&= \frac{1}{t} \left[\sum_{n=0}^{+\infty} (-1)^n \frac{t^{n+1}}{n+1} + \sum_{n=0}^{+\infty} \frac{t^{n+1}}{n+1} \right] \\
&= \sum_{n=0}^{+\infty} \frac{2}{2n+1} t^{2n}.
\end{aligned}$$

By equating the coefficients of t^{2n}, we find:

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1},$$

□

CHAPTER 2

STURM-LIOUVILLE PROBLEM WITH TWO-POINT BOUNDARY CONDITIONS

In this chapter, we study a spectral problem for Sturm-Liouville operator with two-point boundary conditions. We introduce some necessary properties regarding: regular boundary conditions and biorthogonal systems in a Hilbert space. We make a detailed study of the completeness property and the fundamental property of the system of root functions. Finally, we introduce the non-self-adjoint spectral problem that we use in Chapter 3 and which is a key element of a paper published in an international journal [40].

2.1 Sturm-Liouville problem with two-point boundary conditions

We study the linear Sturm-Liouville problem consisting of the equation

$$u''(x) + \lambda u(x) = 0,\ 0 < x < 1,\ \lambda \in \mathbb{C}, \tag{2.1}$$

and the linear two-point boundary conditions of the general form

$$\begin{cases} B_1(u) = a_{11}u'(0) + a_{12}u'(1) + a_{13}u(0) + a_{14}u(1) = 0, \\ B_2(u) = a_{21}u'(0) + a_{22}u'(1) + a_{23}u(0) + a_{24}u(1) = 0, \end{cases} \tag{2.2}$$

where $B_1(u)$ and $B_2(u)$ are linearly independent forms with arbitrary complex-valued coefficients.

We consider the linear operator $\mathcal{L}u = -u''$ defined on $L^2(0,1)$ with the domain

$$D(\mathcal{L}) := \left\{ u \in L^2(0,1) : B_1(u) = B_2(u) = 0 \right\}.$$

Definition 2.1 ([31, Page 193]). A number λ_0 is called an eigenvalue of the operator $\mathcal{L}$ if there exists a function $u^0 \in D(\mathcal{L})$ with $u^0 \neq 0$ such that

$$\mathcal{L}u^0 = \lambda_0 u^0. \tag{2.3}$$

The function u^0 is called the eigenfunction, of the operator $\mathcal{L}$, for the eigenvalue λ_0.

Definition 2.2 ([31, Page 193]). A function u_m is called an associated function of the operator $\mathcal{L}$ of order m $(m = 1, 2, \ldots)$ corresponding to the same eigenvalue λ_0 and the eigenfunction u^0 if satisfies the equation

$$\mathcal{L}u^m = \lambda_0 u^m + u^{m-1}. \tag{2.4}$$

The set of functions $\{u^0, u^1, \ldots\}$ is called the eigen-and associated functions of the problem (2.1)-(2.2).

From the condition (2.2), we can form the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

We denote by $A(ij)$ the matrix composed of the i-th and j-th columns of A, and denote

$$A_{ij} := det A(ij) = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}, \ 1 \leq i < j \leq 4.$$

Then the general solution of equation (2.1) is given by:

$$u(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x), \ \lambda = \mu^2,$$

where c_1 and c_2 are arbitrary constants. Substituting the general solution into the boundary conditions (2.2) for finding c_1 and c_2, we obtain the system of equations

$$\begin{cases} \left(-a_{12}\sin\mu + \frac{a_{13}}{\mu} + a_{14}\frac{\cos\mu}{\mu}\right)c_1 + \left(a_{11} + a_{12}\cos\mu + a_{14}\frac{\sin\mu}{\mu}\right)c_2 = 0, \\ \left(-a_{22}\sin\mu + \frac{a_{23}}{\mu} + a_{24}\frac{\cos\mu}{\mu}\right)c_1 + \left(a_{21} + a_{22}\cos\mu + a_{24}\frac{\sin\mu}{\mu}\right)c_2 = 0. \end{cases} \tag{2.5}$$

Hence, the boundary value problem (2.1)-(2.2) has a nonzero solution if and only if the system (2.5) has a nonzero solution. The eigenvalues of the boundary value problem (2.1)-(2.2) are the roots of the characteristic determinant

$$\Delta(\mu) = \begin{vmatrix} -a_{12}\sin\mu + \frac{a_{13}}{\mu} + a_{14}\frac{\cos\mu}{\mu} & a_{11} + a_{12}\cos\mu + a_{14}\frac{\sin\mu}{\mu} \\ -a_{22}\sin\mu + \frac{a_{23}}{\mu} + a_{24}\frac{\cos\mu}{\mu} & a_{21} + a_{22}\cos\mu + a_{24}\frac{\sin\mu}{\mu} \end{vmatrix}$$

Simple calculations show that

$$\Delta(\mu) = -A_{13} - A_{24} + A_{34}\frac{\sin\mu}{\mu} - (A_{23} + A_{14})\cos\mu + A_{12}\mu\sin\mu. \tag{2.6}$$

Definition 2.3. Two-point boundary conditions (2.2) under one of three conditions

$$\begin{aligned} &(1)\ A_{12} \neq 0, \\ &(2)\ A_{12} = 0, A_{14} + A_{23} \neq 0, \\ &(3)\ A_{12} = 0, A_{14} + A_{23} = 0, A_{34} \neq 0. \end{aligned} \tag{2.7}$$

are called the non-degenerate boundary conditions. Accordingly, if

$$A_{12} = A_{14} + A_{23} = A_{34} = 0, \tag{2.8}$$

then the two-point boundary conditions (2.2) are called the degenerate boundary conditions.

Lemma 2.1 ([31]). *Let the two-point boundary conditions (2.2) be non-degenerate, that is, one of the three conditions (2.7) holds. Then, the problem (2.1)-(2.2) has an infinite countable number of eigenvalues.*

Example 2.1 ([31, page 196]). Consider the Sturm-Liouville problem

$$\begin{cases} -u''(x) = \lambda u(x), \ 0 < x < 1, \\ u'(1) - \alpha u(1) = 0, \ u(0) = 0, \end{cases} \tag{2.9}$$

where $\alpha \in \mathbb{C}$ is a fixed number. From the boundary conditions of the Sturm-Liouville problem (2.9), we have the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It is easy to see that for all α the case (2) from (2.7) holds: $A_{12} = 0$, $A_{14} + A_{23} = 1$. Therefore, the boundary conditions of the Sturm-Liouville problem (2.9) are non-degenerate.

Example 2.2 ([31, page 195]). We consider the Sturm-Liouville problem

$$\begin{cases} -u''(x) = \lambda u(x), \ 0 < x < 1, \\ u'(0) + \alpha u'(1) = 0, \ u(0) - \alpha u(1) = 0, \end{cases} \tag{2.10}$$

where $\alpha \in \mathbb{C}$ is a fixed number. From the boundary conditions of the Sturm-Liouville problem (2.10), we have the matrix

$$A = \begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 0 & 1 & -\alpha \end{bmatrix}$$

It is easy to see that for all α, (2.8) holds: $A_{12} = A_{14} + A_{23} = A_{34} = 0$. Therefore, the boundary conditions of the Sturm-Liouville problem (2.10) are degenerate. From (2.10), we have

$$\Delta(\mu) = -A_{13} - A_{24} = -1 + \alpha^2.$$

Then we obtain that for $\alpha^2 \neq 1$ the Sturm-Liouville problem (2.10) does not have eigenvalues, and for $\alpha^2 = 1$ each number $\lambda \in \mathbb{C}$ is an eigenvalue of this problem.

2.2 Regular boundary conditions

In this section, we present the concept of regular boundary conditions. This concept was first introduced by G. D. Birkhoff in his works in 1908 in [9], for n-th order general ordinary differential operators

$$u^{(n)}(x) + q_2(x) u^{(n-2)}(x) + \ldots + q_{n-1}(x) u'(x) + q_n(x) u(x) = \lambda u(x), \ 0 < x < 1, \tag{2.11}$$

with n linearly independent boundary conditions of the general form

$$B_j(u) = \sum_{s=0}^{n-1} \left(a_{js} u^{(s)}(0) + b_{js} u^{(s)}(1) \right) = 0,\ j = 1, \ldots, n,$$

Rewrite the limit forms $B_j(u)$ by the form

$$a_j u^{(k_j)}(0) + b_j u^{(k_j)}(1) + \sum_{s=0}^{k_j-1} \left(a_{js} u^{(s)}(0) + b_{js} u^{(s)}(1) \right) = 0,\ j = 1, \ldots, n, \tag{2.12}$$

where $|a_j| + |b_j| > 0$, $n - 1 \geq k_1 \geq k_2 \geq \ldots \geq k_n$, $k_j > k_{j+2}$.

Definition 2.4 ([31, page 197]). We denote by $\varepsilon_j = exp\left(i\frac{2\pi j}{n}\right)$, $j = 1, \ldots, n$, the roots of order n from 1.

☞ In the odd case $n = 2m - 1$, the "normed" boundary conditions (2.12) are called the *regular boundary conditions* if the numbers θ_0 and θ_1 defined by the equality

$$\theta_0 + \theta_1 s = \begin{vmatrix} a_1\varepsilon_1^{k_1} & \ldots & a_1\varepsilon_{m-1}^{k_1} & (a_1 + sb_1)\varepsilon_m^{k_1} & b_1\varepsilon_{m+1}^{k_1} & \ldots & b_1\varepsilon_n^{k_1} \\ a_2\varepsilon_1^{k_2} & \ldots & a_2\varepsilon_{m-1}^{k_2} & (a_2 + sb_2)\varepsilon_m^{k_2} & b_2\varepsilon_{m+1}^{k_2} & \ldots & b_2\varepsilon_n^{k_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n\varepsilon_1^{k_n} & \ldots & a_n\varepsilon_{m-1}^{k_n} & (a_n + sb_n)\varepsilon_m^{k_n} & b_n\varepsilon_{m+1}^{k_n} & \ldots & b_n\varepsilon_n^{k_n} \end{vmatrix}$$

are different from zero.

☞ In the even case $n = 2m$, the "normed" boundary conditions (2.12) are called the *regular boundary conditions* if the numbers θ_1 and θ_2 defined by the equality

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} + = \begin{vmatrix} a_1\varepsilon_1^{k_1} & \ldots & a_1\varepsilon_{m-1}^{k_1} & (a_1 + sb_1)\varepsilon_m^{k_1} & \left(a_1 + \frac{b_1}{s}\right)\varepsilon_{m+1}^{k_1} & b_1\varepsilon_{m+2}^{k_1} & \ldots & b_1\varepsilon_n^{k_1} \\ a_2\varepsilon_1^{k_2} & \ldots & a_2\varepsilon_{m-1}^{k_2} & (a_2 + sb_2)\varepsilon_m^{k_2} & \left(a_2 + \frac{b_2}{s}\right)\varepsilon_{m+1}^{k_2} & b_2\varepsilon_{m+2}^{k_2} & \ldots & b_2\varepsilon_n^{k_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n\varepsilon_1^{k_n} & \ldots & a_n\varepsilon_{m-1}^{k_n} & (a_n + sb_n)\varepsilon_m^{k_n} & \left(a_n + \frac{b_n}{s}\right)\varepsilon_{m+1}^{k_n} & b_n\varepsilon_{m+2}^{k_n} & \ldots & b_n\varepsilon_n^{k_n} \end{vmatrix}$$

are different from zero.

An important subclass of the regular boundary conditions, the so-called *strengthened regular boundary conditions* was defined.

Definition 2.5. ☞ In the odd case $n = 2m - 1$ of equation (2.11) all the regular boundary conditions are strengthened regular.

☞ In the even case $n = 2m$ of equation (2.11), we have

1. If $\theta_0^2 - 4\theta_1\theta_2 \neq 0$, the regular boundary conditions (2.12) are called strengthened regular.
2. If $\theta_0^2 - 4\theta_1\theta_2 = 0$, the regular boundary conditions (2.12) are not strengthened regular.

For the case of the Sturm-Liouville problem (2.1)-(2.2), we have $n = 2$, $m = 1$, $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$.

Let first $A_{12} \neq 0$. In this case the boundary conditions (2.2) have the normed form. we have $a_1 = a_{11}$, $b_1 = a_{12}$, $a_2 = a_{21}$, $b_2 = a_{22}$, $k_1 = k_2 = 1$. We calculate the determinant

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} = \begin{vmatrix} (a_1 + sb_1)\,\varepsilon_1^{k_1} & \left(a_1 + \frac{b_1}{s}\right)\varepsilon_2^{k_1} \\ (a_2 + sb_2)\,\varepsilon_1^{k_2} & \left(a_2 + \frac{b_2}{s}\right)\varepsilon_2^{k_2} \end{vmatrix} = \begin{vmatrix} -(a_{11} + sa_{12}) & \left(a_{11} + \frac{a_{12}}{s}\right) \\ -(a_{21} + sa_{22}) & \left(a_{21} + \frac{a_{22}}{s}\right) \end{vmatrix} = A_{12}\left(s - \frac{1}{s}\right).$$

Then $\theta_0 = 0$, $\theta_1 = A_{12}$, $\theta_2 = -A_{12}$. In this case the boundary conditions (2.2) are regular. Since $\theta_0^2 - 4\theta_1\theta_2 = 4A_{12}^2 \neq 0$, in this case the conditions are strengthened regular.

Let now $A_{12} = 0$, and $|a_{11}| + |a_{12}| > 0$. Then the boundary conditions (2.2) can be reduced to the normed form

$$\begin{cases} a_{11}u'(0) + a_{12}u'(1) + \quad a_{13}u(0) + a_{14}u(1) = 0, \\ \qquad\qquad\qquad\qquad\quad a_{23}u(0) + a_{24}u(1) = 0. \end{cases} \tag{2.13}$$

we have $a_1 = a_{11}$, $b_1 = a_{12}$, $a_2 = a_{23}$, $b_2 = a_{24}$, $k_1 = 1$, $k_2 = 0$. We calculate the determinant

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} = \begin{vmatrix} -(a_{11} + sa_{12}) & a_{11} + \frac{a_{12}}{s} \\ a_{23} + sa_{24} & a_{23} + \frac{a_{24}}{s} \end{vmatrix} = -\left(s + \frac{1}{s}\right)(A_{14} + A_{23}) - 2(A_{13} + A_{24}).$$

Then, $\theta_1 = \theta_2 = -(A_{14} + A_{23})$, $\theta_0 = -2(A_{13} + A_{24})$. That is, in this case the boundary conditions (2.13) are regular under the additional condition $A_{14} + A_{23} \neq 0$. The condition of the strengthened regularity will be written in the form:

$$\theta_0^2 - 4\theta_1\theta_2 = (A_{13} + A_{24})^2 - (A_{14} + A_{23})^2 \neq 0.$$

Consider the remaining case $A_{12} = 0$, with $a_{11} = a_{12} = 0$. Then the boundary conditions (2.2) can be reduced to the normed form

$$\begin{cases} a_{13}u(0) + a_{14}u(1) = 0, \\ a_{23}u(0) + a_{24}u(1) = 0. \end{cases} \tag{2.14}$$

We have $a_1 = a_{13}$, $b_1 = a_{14}$, $a_2 = a_{23}$, $b_2 = a_{24}$, $k_1 = k_2 = 0$. We calculate the determinant

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} = \begin{vmatrix} a_{13} + sa_{14} & a_{13} + \frac{a_{14}}{s} \\ a_{23} + sa_{24} & a_{23} + \frac{a_{24}}{s} \end{vmatrix} = A_{34}\left(\frac{1}{s} - s\right).$$

then, $\theta_2 = -\theta_1 = A_{34}$, $\theta_0 = 0$. The inequality $A_{34} \neq 0$ is satisfied in view of the linear independence of the boundary conditions (2.14). Hence, in this case the boundary conditions (2.14) are regular. Since $\theta_0^2 - 4\theta_1\theta_2 = 4A_{34}^2$, these boundary conditions are strengthened regular.

Theorem 2.1 ([31, Theorem 3.105]). *The boundary conditions* (2.2) *are regular in the following three cases:*

$$\begin{aligned} &(1)\ A_{12} \neq 0, \\ &(2)\ A_{12} = 0,\ A_{14} + A_{23} \neq 0, \\ &(3)\ A_{12} = A_{13} = A_{14} = A_{23} = A_{24} = 0,\ A_{34} \neq 0. \end{aligned} \tag{2.15}$$

Here, the boundary conditions will be strengthened regular in the cases (1) *and* (3), *and in the case* (2) *under the additional condition*

$$A_{13} + A_{24} \neq \pm (A_{14} + A_{23}). \tag{2.16}$$

Corollary 2.1 ([31, Corollary 3.106]).

1. *For the case of the Sturm-Liouville equation* (2.1), *all the regular boundary conditions* (2.2) *are non-degenerate.*
2. *Here, the boundary conditions can be non-degenerate and simultaneously irregular in the case when*

$$A_{12} = A_{14} + A_{23} = 0,\ A_{34} \neq 0,\ \textit{and}\ |A_{13}| + |A_{14}| + |A_{23}| + |A_{24}| > 0$$

Example 2.3 ([31, Example 3.107]). Consider the spectral problem

$$\begin{cases} -u''(x) = \lambda u(x), \ 0 < x < 1, \\ u'(0) - \alpha u(1) = 0, \ u(0) = 0, \end{cases} \tag{2.17}$$

where $\alpha \in \mathbb{C}$ is a fixed number. It is easy to see that for all $\alpha \neq 0$ in case (3) from (2.7) holds: $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} = \alpha \neq 0$. Therefore, the boundary conditions of the problem (2.17) are non-degenerate.

Here, the determinant $A_{13} = 1$ is not equal to zero since condition (3) from (2.15) does not hold. Hence, the boundary conditions of the problem (2.17) are not regular.

For convenience of use we reformulate Theorem 2.1 in terms of coefficients of the boundary conditions (2.2).

Theorem 2.2 ([31, Theorem 3.108]). *The boundary conditions* (2.2) *are regular, if one of the following three conditions holds:*

$$\begin{aligned} &(1) \ a_{11}a_{22} - a_{12}a_{21} \neq 0, \\ &(2) \ a_{11}a_{22} - a_{12}a_{21} = 0, \ |a_{11}| + |a_{12}| > 0, \ a_{11}a_{24} + a_{12}a_{23} \neq 0, \\ &(3) \ a_{11} = a_{12} = a_{21} = a_{22} = 0, \ a_{13}a_{24} - a_{14}a_{23} \neq 0. \end{aligned} \tag{2.18}$$

The regular boundary conditions are strengthened regular in the first and third cases, and in the second case under the additional condition

$$a_{11}a_{23} + a_{12}a_{24} \neq a_{11}a_{24} + a_{12}a_{23} \tag{2.19}$$

2.3 Regular but not strengthened regular boundary conditions

From Theorem 2.1 and (2.19), the boundary conditions (2.2) are called regular but not strengthened regular boundary conditions, if the following conditions are hold:

$$A_{12} = 0, \ A_{14} + A_{23} \neq 0, \ A_{13} + A_{24} = \pm (A_{14} + A_{23}).$$

From Theorem 2.2, the regular but not strengthened regular boundary conditions (2.2) can be written in the form

$$\begin{cases} a_{11}u'(0) + a_{12}u'(1) + & a_{13}u(0) + a_{14}u(1) = 0, \\ & a_{23}u(0) + a_{24}u(1) = 0, \end{cases} \tag{2.20}$$

when $|a_{11}| + |a_{22}| > 0$ and two conditions

$$a_{11}a_{24} + a_{12}a_{23} \neq 0, \tag{2.21}$$

$$a_{11}a_{23} + a_{12}a_{24} = \pm (a_{11}a_{24} + a_{12}a_{23}), \tag{2.22}$$

simultaneously hold. Indeed, condition (2.22) can be written in the form:

$$(a_{11} \pm a_{12})(a_{23} \pm a_{24}) = 0.$$

Theorem 2.3 ([42]). *If the boundary conditions* (2.2) *are regular but not strengthened regular, they can be always reduced to the form* (2.20) *(with* $|a_{11}| + |a_{22}| > 0$*) of one of the following four types:*

$$\begin{aligned} &(1)\ a_{11} = a_{12}, \quad a_{23} \neq -a_{24}; \\ &(2)\ a_{11} = -a_{12}, \quad a_{23} \neq a_{24}; \\ &(3)\ a_{23} = a_{24}, \quad a_{11} \neq -a_{12}; \\ &(4)\ a_{23} = -a_{24}, \quad a_{11} \neq a_{12} \end{aligned} \tag{2.23}$$

Corollary 2.2 ([31, Corollary 3.110]). *All regular, but not strengthened regular boundary conditions can be reduced* *to one of the four forms:*

$$\begin{cases} u'(0) - u'(1) + \quad au(0) + bu(1) = 0, \\ \qquad\qquad u(0) + \alpha u(1) = 0, \end{cases} \quad \begin{cases} u'(0) + u'(1) + \quad au(0) + bu(1) = 0, \\ \qquad\qquad u(0) - \alpha u(1) = 0, \end{cases}$$

$$\begin{cases} u'(0) + \alpha u'(1) + \quad au(0) + bu(1) = 0, \\ \qquad\qquad u(0) - u(1) = 0, \end{cases} \quad \begin{cases} u'(0) - \alpha u'(1) + \quad au(0) + bu(1) = 0, \\ \qquad\qquad u(0) + u(1) = 0, \end{cases}$$

where $\alpha \neq 1$, and the coefficients a and b can be arbitrary. For $\alpha = 1$, these boundary conditions are degenerate and consequently, are not regular.

2.4 Biorthogonal systems in Hilbert spaces

Definition 2.6. Let H be a Hilbert space. Two systems of elements (x_k) and (y_k) are said to be biorthogonal systems in H if the relation

$$\langle x_i, y_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{2.24}$$

holds for all values of the indices i and j. Here δ_{ij} is the Kronecker delta.

Definition 2.7. A system of elements of a Hilbert space H is said to be a *complete system* if any vector orthogonal to all vectors of this system is equal to zero.

Definition 2.8. A system of elements of a Hilbert space H is said to be a Riesz basis in H if there exist two constants $m, M > 0$ such that for any $f \in H$, the following inequality holds:

$$m \|f\|_H^2 \leq \sum_{i=0}^{+\infty} f_i^2 \leq M \|f\|_H^2 \tag{2.25}$$

2.5 A non-self-adjoint boundary value problem

In this section we consider main properties of eigen- and associated functions of a non-self-adjoint boundary value problem for a second-order ordinary differential operator.

In $L^2(0,1)$, we consider the operator $\mathcal{L}$ given by:

$$\mathcal{L}X = -X''(x) = \lambda X(x), \; 0 < x < 1, \tag{2.26}$$

and nonlocal boundary conditions:

$$\begin{cases} B_1(X) = X(0) - X(1) = 0, \\ B_2(X) = \beta X'(0) - X'(1) = 0, \; |\beta| < 1. \end{cases} \tag{2.27}$$

where $B_1(X)$ and $B_2(X)$ are linearly independent forms. It is easy to justify that the operator $\mathcal{L}$ is a linear operator on $L^2(0,1)$ defined by (2.26) with the domain

$$D(\mathcal{L}) = \left\{X \in L^2(0,1) : B_1(X) = B_2(X) = 0\right\}.$$

Remark 2.1. From case (3) in Theorem 2.3, the boundary conditions in (2.27) are regular but not strengthened regular.

Proposition 2.1. *The adjoint problem of the boundary value problem* (2.26)-(2.27) *is given by:*

$$\mathcal{L}^*Y = -Y''(x) = \lambda Y(x), \; 0 < x < 1, \tag{2.28}$$

with nonlocal boundary conditions

$$\begin{cases} B_1^*(Y) = Y(0) - \beta Y(1) = 0, \; |\beta| < 1, \\ B_2^*(Y) = Y'(x) - Y'(1) = 0. \end{cases} \tag{2.29}$$

Proof. The operator $\mathcal{L}^*$ defined by (2.28) is a linear operator on $L^2(0,1)$ with the domain

$$D(\mathcal{L}^*) = \left\{X \in L^2(0,1) : B_1^*(X) = B_2^*(X) = 0\right\}.$$

Let $X \in D(\mathcal{L})$ and $Y \in D(\mathcal{L}^*)$, by integration by parts twice, we obtain

$$\begin{aligned}\langle \mathcal{L}X, Y\rangle &= -\int_0^1 X''(x)\, Y(x)\, dx \\ &= X'(0)\left[Y(0) - \beta Y(1)\right] + X(0)\left[Y'(1) - Y'(0)\right] - \int_0^1 X(x)\, Y''(x)\, dx \\ &= \langle X, \mathcal{L}^* Y\rangle,\end{aligned}$$

then, $\mathcal{L}^*$ is the adjoint operator of the operator $\mathcal{L}$. Since, the problem (2.28)-(2.29) is the adjoint problem of the problem (2.26)-(2.27). □

Remark 2.2. Obviously $D(\mathcal{L}) \neq D(\mathcal{L}^*)$ then, the spectral problems (2.26)-(2.27) and (2.28)-(2.29) are not self-adjoint.

Proposition 2.2. *We have*

1. *The two spectral problem (2.26)-(2.27) and (2.28)-(2.29) have the same double eigenvalues $\lambda_k = (2\pi k)^2$ (except for the first $\lambda_0 = 0$). The set of eigenfunctions of the problems (2.26)-(2.27) and (2.28)-(2.29) are the following:*

$$X_0(x) = b_0;\; X_{2k-1}(x) = A_k \cos(2\pi k x),\; k \in \mathbb{N}^*,\; b_0,\; A_k \in \mathbb{R}, \tag{2.30}$$

$$Y_0(x) = a_0'\left(x + \frac{\beta}{1-\beta}\right);\; Y_{2k}(x) = B_k' \sin(2\pi k x),\; k \in \mathbb{N}^*,\; a_0',\; B_k' \in \mathbb{R}. \tag{2.31}$$

2. *The sets of eigenfunctions $\{X_0(x), X_{2k-1}(x)\}$ and $\{Y_0(x), Y_{2k}(x)\}$, $k \in \mathbb{N}^*$, are not complete in the space $L^2(0,1)$.*

Proof. 1. We have:

(a) If $\lambda = 0$ in (2.26) and (2.28), then $X_0(x) = a_0 x + b_0$ and $Y_0(x) = a_0' x + b_0'$. From (2.27), we obtain:

$$\begin{cases} X(0) = X(1), \\ \beta X'(0) = X'(1) \end{cases} \Leftrightarrow \begin{cases} b_0 = a_0 + b_0 \\ \beta a_0 = \beta a_0 \end{cases} \Leftrightarrow a_0 = 0,$$

then $X_0(0) = b_0$. On other hand,

$$\begin{cases} Y(0) = \beta Y(1), \\ Y'(0) = Y'(1) \end{cases} \Leftrightarrow \begin{cases} b_0' = \beta(a_0' + b_0') \\ a_0' = a_0' \end{cases} \Leftrightarrow b_0' = \frac{\beta a_0'}{1-\beta},$$

then, $Y_0(x) = a_0'\left(x + \frac{\beta}{1-\beta}\right)$.

(b) If $\lambda = \mu^2$, the general solutions of equations (2.26) and (2.28) are given by :

$$X_{2k-1}(x) = A_k \cos(\mu x) + B_k \sin(\mu x), \ Y_{2k}(x) = A_k' \cos(\mu x) + B_k' \sin(\mu x), \ \ A_k, B_k, A_k', B_k' \in \mathbb{R}.$$

From boundary conditions (2.27), we have:

$$\begin{cases} X_{2k-1}(0) = X_{2k-1}(1), \\ \beta X_{2k-1}'(0) = X_{2k-1}'(1). \end{cases} \Leftrightarrow \begin{cases} A_k(\cos\mu - 1) + B_k \sin\mu = 0, \\ -A_k \sin\mu + B_k(\cos\mu - \beta) = 0. \end{cases} \tag{2.32}$$

The system (2.32) admits a non-trivial solution, then the determinant of this system is zero. Therefore, we have:

$$\Delta(\mu) = \begin{vmatrix} \cos\mu - 1 & \sin\mu \\ -\sin\mu & \cos\mu - \beta \end{vmatrix} = 2(\beta+1)\sin^2(\mu/2) = 0 \Leftrightarrow \mu_k = 2\pi k, \ k \in \mathbb{N}^*,$$

then $\lambda_k = (2\pi k)^2$ are multiple eigenvalues and From (2.32), we obtain $B_k = 0$. Then, $X_{2k-1} = A_k \cos(2\pi k x)$ are eigenfunctions.

From boundary conditions (2.29), we have:

$$\begin{cases} Y_{2k}(0) = \beta Y_{2k}(1), \\ Y_{2k}'(0) = Y_{2k}'(1). \end{cases} \Leftrightarrow \begin{cases} A_k'(\beta\cos\mu - 1) + B_k' \sin\mu = 0, \\ -A_k' \sin\mu + B_k'(\cos\mu - 1) = 0. \end{cases} \tag{2.33}$$

The system (2.33) admits a non-trivial solution, then the determinant of this system is zero.

Therefore, we have:

$$\Delta(\mu)=\begin{vmatrix}\beta\cos\mu-1 & \sin\mu\\ -\sin\mu & \cos\mu-1\end{vmatrix}=2\left(2-(\beta-1)\cos\mu\right)\sin^2(\mu/2)=0\Leftrightarrow\mu_k=2\pi k,\ k\in\mathbb{N},$$

then $\lambda_k=(2\pi k)^2$ are multiple eigenvalues and from (2.33), we obtain $A'_k=0$. Then, $Y_{2k}=B'_k\sin(2\pi kx)$ are eigenfunctions.

(c) If $\lambda=-\mu^2$, with $\mu\neq 0$, the general solutions of equations (2.26) and (2.28) are given by :

$$X_{2k}(x)=C_ke^{\mu x}+D_ke^{-\mu x},\ Y_{2k-1}=C'_ke^{\mu x}+D'_ke^{-\mu x},\ C_k,D_k,C'_k,D'_k\in\mathbb{R}.$$

From boundary conditions (2.27), we have:

$$\begin{cases}X_{2k}(0)=X_{2k}(1),\\ \beta X'_{2k}(0)=X'_{2k}(1).\end{cases}\Leftrightarrow\begin{cases}(e^{\mu}-1)C_k+(e^{-\mu}-1)D_k=0,\\ (e^{\mu}-\beta)C_k+(\beta-e^{-\mu})D_k=0.\end{cases}\tag{2.34}$$

The system (2.34) admits a non-trivial solution, then the determinant of this system is zero. Therefore, we have:

$$\Delta(\mu)=\begin{vmatrix}e^{\mu}-1 & e^{-\mu}-1\\ e^{\mu}-\beta & \beta-e^{-\mu}\end{vmatrix}=2(\beta+1)(\cosh\mu-1)=0,$$

then, $\mu=0$ (impossible).

From boundary conditions (2.29), we have:

$$\begin{cases}Y_{2k-1}(0)=\beta Y_{2k-1}(1),\\ Y'_{2k-1}(0)=Y'_{2k-1}(1).\end{cases}\Leftrightarrow\begin{cases}(\beta e^{\mu}-1)C'_k+(\beta e^{-\mu}-1)D'_k=0,\\ (e^{\mu}-1)C'_k+(1-e^{-\mu})D'_k=0.\end{cases}\tag{2.35}$$

The system (2.35) admits a non-trivial solution, then the determinant of this system is zero. Therefore, we have:

$$\Delta(\mu)=\begin{vmatrix}\beta e^{\mu}-1 & \beta e^{-\mu}-1\\ e^{\mu}-1 & 1-e^{-\mu}\end{vmatrix}=2(\beta+1)(\cosh\mu-1)=0,$$

then, $\mu = 0$ (impossible).

2. Let $f(x) = 2\sin(2\pi kx)$ and $g(x) = 2\cos(2\pi kx)$. We have:

$$\langle X_{2k-1}(x), f(x)\rangle = \int_0^1 2a_k \cos(2\pi kx)\sin(2\pi kx)\,dx = \int_0^1 a_k \sin(4\pi kx)\,dx = \frac{a_k}{4\pi k}\left[\cos(4\pi kx)\right]_0^1 = 0,$$

and

$$\langle Y_{2k}(x), g(x)\rangle = \int_0^1 2a'_k \sin(2\pi kx)\cos(2\pi kx)\,dx = \int_0^1 a'_k \sin(4\pi kx)\,dx = \frac{a'_k}{4\pi k}\left[\cos(4\pi k)\right]_0^1 = 0,$$

then, sets of eigenfunctions $\{X_0, X_{2k-1}(x)\}$ and $\{Y_0(x), Y_{2k}(x)\}$, $k \in \mathbb{N}^*$, are not complete in $L^2(0,1)$.

□

To make the set $\{X_0(x), X_{2k-1}(x)\}$, $k \in \mathbb{N}^*$, a complete set on $L^2(0,1)$, we have to look for the associated eigenfunctions $X_{2k}(x)$. According to Definition 2.2, we have to solve the following spectral problem:

$$\begin{cases} -X''_{2k}(x) = \lambda_k X_{2k}(x) + X_{2k-1}(x),\ 0 < x < 1, \\ X_{2k}(0) = X_{2k}(1),\ \beta X'_{2k}(0) = X'_{2k}(1). \end{cases} \tag{2.36}$$

The solution of spectral problem (2.36) is given by:

$$X_{2k}(x) = \left(C_k + \frac{A_k}{16\pi^2 k^2}\right)\cos(2\pi kx) + \frac{A_k}{4\pi k}\left(\frac{1}{\beta - 1} + x\right)\sin(2\pi kx),$$

where $A_k, C_k \in \mathbb{R}$ and $k \in \mathbb{N}^*$. Similarly for the set $\{Y_0(x), Y_{2k}(x)\}$, $k \in \mathbb{N}^*$, we have to solve the following spectral problem:

$$\begin{cases} -Y''_{2k-1}(x) = \lambda_k Y_{2k-1}(x) + Y_{2k}(x),\ 0 < x < 1, \\ Y_{2k-1}(0) = \beta Y_{2k-1}(1),\ Y'_{2k-1}(0) = Y'_{2k-1}(1). \end{cases} \tag{2.37}$$

The solution of spectral problem (2.37) is given by:

$$Y_{2k-1}(x) = \frac{B_k}{4\pi k}\left(\frac{\beta}{\beta - 1} - x\right)\cos(2\pi kx) + \left(D_k + \frac{B_k}{16\pi^2 k^2}\right)\sin(2\pi kx),$$

where $B_k, D_k \in \mathbb{R}$ and $k \in \mathbb{N}^*$.

To show that the two systems of functions $\{X_0(x), X_{2k-1}(x), X_{2k}(x)\}$ and $\{Y_0(x), Y_{2k-1}(x), Y_{2k}(x)\}$ are biorthogonal, we must explicitly determine the coefficients b_0, a_0, A_k, B_k, C_k and D_k. According the biorthogonality condition:

$$\langle X_0, Y_0\rangle = 1,\ \langle X_{2k-1}, Y_{2k-1}\rangle = 1,\ \langle X_{2k}, Y_{2k}\rangle = 1,\ \langle X_{2k}, Y_{2k-1}\rangle = 0. \tag{2.38}$$

From the first condition we get

$$1 = \langle X_0, Y_0\rangle = b_0 a_0' \int_0^1 \left(x + \frac{\beta}{1-\beta}\right) dx = b_0 a_0' \left[\frac{x^2}{2} + \frac{\beta x}{1-\beta}\right]_0^1 = b_0 a_0' \frac{(1+\beta)}{2(1-\beta)}.$$

Then, $b_0 = 2$ and $a_0' = \frac{1-\beta}{1+\beta}$. In this case, we have:

$$X_0(x) = 2,\ Y_0(x) = ax + b, \tag{2.39}$$

where $a = \frac{1-\beta}{1+\beta}$ and $b = \frac{\beta}{1+\beta}$.

We use the second condition from (2.38):

$$\begin{aligned}
1 = \langle X_{2k-1}, Y_{2k-1}\rangle &= \frac{A_k B_k}{4\pi k} \int_0^1 \left(\frac{\beta}{\beta-1} + x\right) \cos^2(2\pi k x)\, dx + \frac{A_k}{2}\left(D_k + \frac{B_k}{16\pi^2 k^2}\right) \int_0^1 \sin(4\pi k x)\, dx \\
&= \frac{A_k B_k}{8\pi k} \int_0^1 \left(\frac{\beta}{\beta-1} - x\right) dx + \frac{A_k B_k}{8\pi k} \int_0^1 \left(\frac{\beta}{\beta-1} - x\right) \cos(4\pi k x)\, dx \\
&= \frac{A_k B_k}{8\pi k} \left[\frac{\beta x}{\beta-1} - \frac{x^2}{2}\right]_0^1 + \frac{A_k B_k}{8\pi k} \left[\left(\frac{\beta}{\beta-1} - x\right) \frac{\sin(4\pi k x)}{4\pi k}\Bigg|_0^1 + \int_0^1 \frac{\sin(4\pi k x)}{4\pi k} dx\right] \\
&= \frac{A_k B_k (\beta+1)}{16\pi k(\beta-1)}.
\end{aligned}$$

Then, $A_k = 4$ and $B_k = \frac{4\pi k(\beta-1)}{1+\beta}$. In this case, we have:

$$X_{2k-1}(x) = 4\cos(2\pi k x),\ Y_{2k-1}(x) = (ax+b)\cos(2\pi k x) + \left(D_k - \frac{a}{4\pi k}\right) \sin(2\pi k x), \tag{2.40}$$

where $D_k \in \mathbb{R}$. We use the third condition from (2.38):

$$\begin{aligned} 1 = \langle X_{2k}, Y_{2k} \rangle &= \frac{B'_k}{2}\left(C_k + \frac{A_k}{16\pi^2 k^2}\right)\int_0^1 \sin(4\pi k x)\,dx + \frac{A_k B'_k}{4\pi k}\int_0^1 \left(\frac{1}{\beta-1} + x\right)\sin^2(2\pi k x)\,dx \\ &= \frac{A_k B'_k}{8\pi k}\int_0^1 \left(\frac{1}{\beta-1} + x\right)dx - \frac{A_k B'_k}{8\pi k}\int_0^1 \left(\frac{1}{\beta-1} + x\right)\cos(4\pi k x)\,dx \\ &= \frac{A_k B'_k}{8\pi k}\left[\frac{x}{\beta-1} + \frac{x^2}{2}\right]_0^1 \\ &= \frac{A_k B'_k}{16\pi k}\frac{\beta+1}{\beta-1}. \end{aligned}$$

Then, $B'_k = 1$ and $A_k = \frac{16\pi k(\beta-1)}{\beta+1}$. In this case, we have:

$$X_{2k}(x) = \left(C_k - \frac{a}{\pi k}\right)\cos(2\pi k x) + 4(1-b-ax)\sin(2\pi k x),\ Y_{2k}(x) = \sin(2\pi k x), \tag{2.41}$$

where $C_k \in \mathbb{R}$. We use the fourth condition from (2.38), and from (2.40), (2.41) we have:

$$\begin{aligned} 0 = \langle X_{2k}, Y_{2k-1} \rangle &= \left(C_k - \frac{a}{\pi k}\right)\int_0^1 (ax+b)\cos^2(2\pi k x)\,dx + 4\left(D_k - \frac{a}{4\pi k}\right)\int_0^1 (1-b-ax)\sin^2(2\pi k x)\,dx \\ &= \left(\frac{C_k}{2} - \frac{a}{2\pi k}\right)\left[\int_0^1 (ax+b)\,dx + \int_0^1 (ax+b)\cos(4\pi k x)\,dx\right] \\ &+ 2\left(D_k - \frac{a}{4\pi k}\right)\left[\int_0^1 (1-b-ax)\,dx - \int_0^1 (1-b-ax)\cos(4\pi k x)\,dx\right] \\ &= a\left(\frac{C_k}{2} - \frac{a}{2\pi k}\right) - 2a\left(D_k - \frac{a}{4\pi k}\right) \\ &= a\left(\frac{C_k}{2} - 2D_k\right). \end{aligned}$$

Then, we have:

$$C_k = 4D_k. \tag{2.42}$$

Thus, from (2.39)-(2.42), the system given by:

$$X_0(x) = 2,\ X_{2k-1}(x) = 4\cos(2\pi k x),\ X_{2k}(x) = 4\left(D_k - \frac{a}{4\pi k}\right)\cos(2\pi k x) + 4(1-b-ax)\sin(2\pi k x) \tag{2.43}$$

will be biorthogonal to the system

$$Y_0(x) = ax + b,\ Y_{2k-1}(x) = (ax+b)\cos(2\pi kx) + \left(D_k - \frac{a}{4\pi k}\right)\sin(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx). \tag{2.44}$$

We calculate the norms of elements of the biorthogonal systems (2.43) and (2.44).

$$\|X_0\| = 2,\ \|X_{2k-1}\| = 2\sqrt{2},\ \|X_{2k}\|^2 = 8\left(D_k - \frac{a}{4\pi k}\right)^2 + \frac{4}{\pi k a}\left(D_k - \frac{a}{4\pi k}\right) - \frac{8}{3a}\left(2b^3 - 3b^2 + 3b - 1\right),$$
$$\|Y_0\| = \sqrt{\frac{a^2+3b^2+3ab}{3}},\ \|Y_{2k-1}\|^2 = \frac{1}{2}\left(D_k - \frac{a}{4\pi k}\right)^2 - \frac{a}{4\pi k}\left(D_k - \frac{a}{4\pi k}\right) + \frac{a^2}{8\pi^2 k^2} + \frac{1}{6}\left(a^2+3ab+3b^2\right),$$
$$\|Y_{2k}\| = \frac{\sqrt{2}}{2}.$$

Using [31, Theorem 3.151], we check for which of the constants D_k the necessary and sufficient condition of the unconditional basis holds.

If the sequence (D_k) is unbounded. Then,

$$\lim_{k\to+\infty} \|X_{2k}\|\,\|Y_{2k}\| = +\infty,\ \lim_{k\to+\infty} \|X_{2k-1}\|\,\|Y_{2k-1}\| = +\infty.$$

Hence, the requirement of the criterion in [44, Theorem 3.151] for an unconditional basis does not hold. Therefore, the systems (2.43) and (2.44) does not give an unconditional basis in $L^2(0,1)$.

In the case when the sequence D_k is bounded, we get

$$\lim_{k\to+\infty} \|X_{2k}\|\,\|Y_{2k}\| < +\infty,\ \lim_{k\to+\infty} \|X_{2k-1}\|\,\|Y_{2k-1}\| < +\infty.$$

Then, we obtain the following result for the systems (2.43) and (2.44).

Lemma 2.2. *If $D_k = \frac{a}{4\pi k}$ with $k \in \mathbb{N}^*$, then the systems*

$$X_0(x) = 2,\ X_{2k-1}(x) = 4\cos(2\pi kx),\ X_{2k} = 4(1-b-ax)\sin(2\pi kx), \tag{2.45}$$

and

$$Y_0 = ax+b,\ Y_{2k-1}(x) = (ax+b)\cos(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx), \tag{2.46}$$

are bi-orthonormal in $L^2(0,1)$.

Proof. It is easy to show that the systems (2.45) and (2.46) form a bi-orthogonal system on $[0,1]$, i.e.

$$\langle X_i, Y_j\rangle = \int_0^1 X_i(x)\, Y_j(x)\, dx = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

□

Lemma 2.3. *The systems of functions* (2.45) *and* (2.46) *are complete in* $L^2(0,1)$.

Proof. Let $f \in L^2(0,1)$ be orthogonal with the system of functions (2.45). $f(x)$ can be presented by the series

$$f(x) = \sum_{n=1}^{+\infty} B_n \sin(2\pi n x), \tag{2.47}$$

which converges in $L^2(0,1)$. Since $f(x)$ is orthogonal with (2.45), we have

$$\begin{aligned} 0 &= \int_0^1 4(1-b-ax)\, f(x) \sin(2\pi k x)\, dx \\ &= \sum_{n=1}^{+\infty} B_n \int_0^1 4(1-b-ax) \sin(2\pi k x) \sin(2\pi n x)\, dx = B_k,\ k \in \mathbb{N}^*. \end{aligned}$$

From (2.47), $f(x) = 0$. Then, (2.45) is complete in $L^2(0,1)$. □

The following theorem is valid:

Theorem 2.4. *The system of functions* (2.45) *forms a Riesz basis in* $L^2(0,1)$.

Proof. From [31, page 211], the system (2.45) is a Riesz basis in $L^2(0,1)$ if there exist two constants $m, M > 0$ such that for any $f \in L^2(0,1)$, the following inequality holds:

$$m \|f\|^2_{L^2(0,1)} \leq \sum_{i=0}^{+\infty} f_i^2 \leq M \|f\|^2_{L^2(0,1)},$$

where

$$f_i = \langle f, Y_i\rangle = \int_0^1 f(x)\, Y_i(x)\ dx \text{ and } \bar{f}_i = \langle f, X_i\rangle = \int_0^1 f(x)\, X_i(x)\ dx. \tag{2.48}$$

For $i=0$, and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} f_0^2 = \langle f, Y_0\rangle^2 = \left[\int_0^1 Y_0(x) f(x)\,dx\right]^2 &\leq \int_0^1 Y_0^2(x)\,dx \int_0^1 f^2(x)\ dx \\ &\leq \frac{1+\beta+\beta^2}{3(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned} \tag{2.49}$$

For $i=2k-1$, and using the Bessel inequality we obtain:

$$\begin{aligned} \sum_{k=1}^{+\infty} f_{2k-1}^2 = \sum_{k=1}^{+\infty}\langle f, Y_{2k-1}\rangle^2 &\leq \|Y_{2k-1}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)} \\ &\leq \frac{7-11\beta+7\beta^2}{6(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned} \tag{2.50}$$

For $i=2k$, and using the Bessel inequality we obtain:

$$\sum_{k=1}^{+\infty} f_{2k}^2 = \sum_{k=1}^{+\infty}\langle f, Y_{2k}\rangle^2 \leq \|Y_{2k}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)} \leq \frac{1}{2}\|f\|^2_{L^2(0,1)}. \tag{2.51}$$

From (2.49)-(2.51), we have

$$\sum_{i=0}^{+\infty} f_i^2 = f_0^2 + \sum_{k=1}^{+\infty} f_{2k-1}^2 + \sum_{k=1}^{+\infty} f_{2k}^2 \leq M\|f\|^2_{L^2(0,1)}, \tag{2.52}$$

where $M=\frac{4-\beta+4\beta^2}{2(1+\beta)^2}$.

On the other hand we have:

$$\bar{f}_0^2 = \langle f, X_0\rangle^2 = \left[\int_0^1 X_0(x) f(x)\ dx\right]^2 \leq 4\|f\|^2_{L^2(0,1)}. \tag{2.53}$$

Using the Bessel inequality, we obtain:

$$\sum_{k=1}^{+\infty} \bar{f}_{2k-1}^2 = \sum_{k=1}^{+\infty}\langle f, X_{2k-1}\rangle^2 \leq 8\|f\|^2_{L^2(0,1)}, \tag{2.54}$$

$$\sum_{k=1}^{+\infty} \bar{f}_{2k}^2 = \sum_{k=1}^{+\infty}\langle f, X_{2k}\rangle^2 \leq \frac{8\left(1+\beta+\beta^2\right)}{3(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \tag{2.55}$$

Then, from (2.53)-(2.55) we have:

$$\sum_{i=0}^{+\infty} \bar{f}_i^2 \leq \frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2}\left\|f\right\|_{L^2(0,1)}^2. \tag{2.56}$$

Using the Cauchy-Schwarz inequality and (2.56), we get

$$\begin{aligned}
\left\|f\right\|_{L^2(0,1)}^2 &= \langle f, f\rangle \\
&= \sum_{i=0}^{+\infty} \bar{f}_i f_i \\
&\leq \left[\sum_{i=0}^{+\infty} \bar{f}_i^2\right]^{1/2} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2} \\
&\leq \left[\frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2}\right]^{1/2} \left\|f\right\|_{L^2(0,1)} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2}.
\end{aligned}$$

Consequently, we have:

$$m\left\|f\right\|_{L^2(0,1)}^2 \leq \sum_{i=0}^{+\infty} f_i^2,\ m = \frac{3\left(1+\beta\right)^2}{44+80\beta+44\beta^2}. \tag{2.57}$$

From (2.52) and (2.57), the system (2.45) is a Riesz basis in $L^2(0,1)$. □

Corollary 2.3. *From Lemma 2.2 and Theorem 2.4, the systems* (2.45) *and* (2.46) *are equivalent bases in* $L^2(0,1)$.

CHAPTER 3

AN INVERSE TIME-DEPENDENT SOURCE PROBLEM

In this chapter, we study the inverse time-dependent source problem (2)- (4) and (5). The peculiarity of this inverse problem is that the system of eigenfunctions is not complete, but the system of eigenfunctions and associated functions forms a basis in $L^2(0,1)$. Under certain natural conditions of regularity and consistency of the input data, the existence, uniqueness, and continuous data dependence of the solution are shown using the generalized Fourier method, Mittag-Leffler function estimates, and the Banach contraction principle. This chapter is a draft of an article titled "*An inverse time-dependent source problem for a time-fractional diffusion equation witn nonlocal boundary conditions*" published in an international journal [40].

3.1 Statement of the problem

In this section, we are interested with an inverse source problem of recovering a time-dependent source term $r(t)$ and $u(x,t)$ for the one-dimensional time-fractional diffusion equation given by (5) such that:

$$ {}^{c}\mathcal{D}_t^{\alpha,\rho} u(x,t) = u_{xx} + r(t) f(x,t), \ (x,t) \in \Omega_T, \tag{3.1} $$

supplemented with the initial condition

$$ u(x,0) = \varphi(x), \ 0 < x < 1, \tag{3.2} $$

and nonlocal family of boundary conditions

$$\begin{cases} u(0,t) = u(1,t), \\ \beta u_x(0,t) = u_x(1,t), \end{cases} \qquad 0 < t \leq T, \tag{3.3}$$

where $-1 < \beta < 1$ and $\Omega_T := \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$ for some fixed $T > 0$, ${}^{c}\mathcal{D}_t^{\alpha,\rho}$ stands for left-sided generalized Caputo fractional derivative of order $0 < \alpha \leq 1$, $\rho > 0$ is a real constant, $\varphi(x)$ and $f(x,t)$ are given functions on $[0,1]$ and $\bar{\Omega}_T$ respectively.

In the physical sense, the first condition in (3.3) means the equality of the distribution densities at the ends of the interval $[0,1]$, and the second condition in (3.3) means the proportionality of fluxes across opposite boundaries, where β is a coefficient characterizing the proportionality of the flux at one end and the rate of change of the average of flux over of the interval $[0,1]$.

The direct problem is to find the solution $u(x,t)$ that satisfies (3.1)-(3.3), when the function $r(t)$ is known. The structure of the source term $r(t) f(x,t)$ in (3.1) arises in microwave heating processes, where $r(t)$ is proportional to the power of the external energy source, and $f(x,t)$ represents the local conversion rate of microwave energy. The external energy is delivered to the target at a controlled level by microwave-generating equipment. The inverse source problem for such a model provides insight into how the total energy content can be externally controlled. However, our focus is on determining the pair of functions $\{u(x,t), r(t)\}$ from (3.1)-(3.3), subject to an integral over-determination condition

$$\int_0^1 u(x,t)\, dx = g(t),\ 0 \leq t \leq T, \tag{3.4}$$

where $g(t)$ is a given function representing the total amount of diffusion in the interval $[0,1]$. The integral condition (3.4) arises naturally and serves as supplementary information for identifying the source term. This type of condition is capable of modeling various physical phenomena in the contexts of chemical engineering, thermo-elasticity, heat conduction and diffusion processes, and fluid flow in porous media [21].

We aim to solve the direct problem (3.1)-(3.3) using the Fourier method, commonly known as the method of separation of variables. The spectral problem associated with the corresponding homogeneous form of (3.1)-(3.3) is given by the boundary value problem (2.26)-(2.27). Recall that this boundary value problem is non-self-adjoint, and the set of eigenfunctions corresponding to the spectral problem (2.26)-

(2.27) is not complete in the space $L^2(0,1)$. We supplement the set of eigenfunctions with associated eigenfunctions to form a complete system in $L^2(0,1)$. Another complete set of eigenfunctions and associated eigenfunctions of the adjoint problem (2.28)-(2.29) was obtained in Proposition 2.1 to construct a biorthogonal system of functions.

A regular solution to the inverse time-dependent source problem is a pair of functions $\{u(x,t), r(t)\}$ such that

$$u(\cdot,t) \in \mathcal{C}^2(0,1), \ {}^c\mathcal{D}_t^{\alpha,\rho} u(x,\cdot) \in \mathcal{C}(0,T) \text{ and } r \in \mathcal{C}(0,T),$$

and which satisfy equations (3.1)-(3.4).

Our strategy is primarily based on Fourier's method, constructing a series solution using a biorthogonal system of functions derived from the eigen-and associated functions (2.45) and (2.46) of the spectral problem (2.26)-(2.27) and its conjugate problem (2.28)-(2.29). Under suitable regularity and consistency conditions on the input data, and by employing estimates of the Mittag-Leffler function along with Banach's contraction mapping principle, we establish the existence, uniqueness, and stability of the solution to the inverse time-dependent source problem (3.1)-(3.4).

3.2 Main results

3.2.1 Existence and uniqueness of the solution

In this subsection, we present the main result on the existence and uniqueness of the solution to the inverse time-dependent source problem (3.1)-(3.4).

Theorem 3.1 ([40]). *Let the following assumptions be satisfied*

(A1) $\varphi \in \mathcal{C}^4(0,1)$, $\varphi(1) = \varphi(0)$, $\varphi'(1) = \beta\varphi'(0)$, $\varphi''(1) = \varphi''(0)$, $\varphi'''(1) = \beta\varphi'''(0)$;

(A2) $f(x,\cdot) \in \mathcal{C}[0,T]$ *and for* $t \in [0,T]$, $f(\cdot,t) \in \mathcal{C}^4[0,1]$; $f(0,t) = f(1,t)$; $f_x(1,t) = \beta f_x(0,t)$; $f_{xx}(0,t) = f_{xx}(1,t)$; $f_{xxx}(1,t) = \beta f_{xxx}(0,t)$; $\int_0^1 f(x,t)\,dx \neq 0$ *and there exists a constant* $M > 0$ *such that*

$$0 < \left| \int_0^1 f(x,t)\,dx \right|^{-1} \leq M;$$

(A3) $g \in \mathcal{C}^1(0,T)$, *and* g *satisfies the consistency condition* $\int_0^1 \varphi(x)\,dx = g(0)$.

If the following condition

$$T < \left(\frac{\alpha \left|1+\beta\right| \rho^{\alpha}}{MC' \left|1-\beta\right|} \right)^{1/\rho\alpha}, \tag{3.5}$$

where C' is defined in (3.22), then the inverse time-dependent problem (3.1)-(3.4) has a unique solution.

Proof. According to assumptions (A1)-(A3), there are positive constants, L_1, L_2, M_i, $i = 0, \ldots, 2$, such that

$$L_1 := \max_{0\leq t\leq T} e_{\alpha}\left(\frac{t^{\rho}}{\rho}, \lambda_k\right),\ L_2 := \max_{0\leq s\leq t\leq T} E_{\alpha,\alpha}\left[-\lambda_k \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right],\ M_0 := \|r\|_{\mathcal{C}(0,T)};$$
$$M_1 := \max\left(\|f_0\|_{\mathcal{C}(0,T)}, \left\|f_{2k-1}^{(4)}\right\|_{\mathcal{C}(0,T)}, \left\|f_{2k}^{(4)}\right\|_{\mathcal{C}(0,T)}\right),\ M_2 := \max\left(|\varphi_0|, \left|\varphi_{2k-1}^{(4)}\right|, \left|\varphi_{2k}^{(4)}\right|\right).$$

The proof of this theorem takes place in three steps:

Step 1: Construction of solution . By applying the Fourier's method, the solution $u(x,t)$ of the direct problem (3.1)-(3.3), can be developed in uniformly convergent series form using the eigenfunctions (2.45) in $L^2(0,1)$ as follows

$$u(x,t) = 2u_0(t) + \sum_{k=1}^{+\infty} u_{2k-1}(t) X_{2k-1}(x) + \sum_{k=1}^{+\infty} u_{2k}(t) X_{2k}(x), \tag{3.6}$$

We define the coefficients $u_0(t)$, $u_{2k-1}(t)$ and $u_{2k}(t)$ for $k \in \mathbb{N}^*$ by multiplying (3.6) by the eigenfunctions of (2.46) and integrating over $[0,1]$ and using Lemma 2.2, we get

$$u_0(t) = \langle u(x,t), Y_0(x)\rangle,\ u_{2k-1}(t) = \langle u(x,t), Y_{2k-1}(x)\rangle,\ u_{2k}(t) = \langle u(x,t), Y_{2k}(x)\rangle, \tag{3.7}$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in $L^2(0,1)$.

The expansion coefficients of the functions $f(x,t)$ and $\varphi(x)$ into eigenfunctions (2.46) are given by

$$f_0(t) = \langle f(x,t), Y_0(x)\rangle,\ f_{2k-1}(t) = \langle f(x,t), Y_{2k-1}(x)\rangle,\ f_{2k}(t) = \langle f(x,t), Y_{2k}(x)\rangle, \tag{3.8}$$

and

$$\varphi_0 = \langle \varphi(x), Y_0(x)\rangle,\ \varphi_{2k-1} = \langle \varphi(x), Y_{2k-1}(x)\rangle,\ \varphi_{2k} = \langle \varphi(x), Y_{2k}(x)\rangle. \tag{3.9}$$

From (3.1), (3.7)-(3.9), Lemma 2.2, integration by parts twice and (3.3), we obtain

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho}u_{0}\left(t\right)=r\left(t\right)f_{0}\left(t\right), \\ u_{0}\left(0\right)=\varphi_{0}, \end{cases} \tag{3.10}$$

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho}u_{2k}\left(t\right)+\lambda_{k}u_{2k}\left(t\right)=r\left(t\right)f_{2k}\left(t\right), \\ u_{2k}\left(0\right)=\varphi_{2k}, \end{cases} \tag{3.11}$$

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho}u_{2k-1}\left(t\right)+\lambda_{k}u_{2k-1}\left(t\right)=-4\pi aku_{2k}\left(t\right)+r\left(t\right)f_{2k-1}\left(t\right), \\ u_{2k-1}\left(0\right)=\varphi_{2k-1}. \end{cases} \tag{3.12}$$

Applying $\mathcal{I}_{0}^{\alpha,\rho}$ on (3.10) and using Theorem 1.3, we obtain

$$u_{0}\left(t\right)=\varphi_{0}+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}r\left(s\right)f_{0}\left(s\right)\frac{ds}{s^{1-\rho}}. \tag{3.13}$$

Applying Theorem 1.7 on (3.11) and (3.12) , we obtain:

$$u_{2k}\left(t\right)=\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)+\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)r\left(s\right)f_{2k}\left(s\right)\frac{ds}{s^{1-\rho}}, \tag{3.14}$$

and

$$\begin{aligned} u_{2k-1}\left(t\right)=&\ \varphi_{2k-1}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)+\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)r\left(s\right)f_{2k-1}\left(s\right)\frac{ds}{s^{1-\rho}} \\ &-4\pi ak\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)u_{2k}\left(s\right)\frac{ds}{s^{1-\rho}}. \end{aligned} \tag{3.15}$$

After substituting expressions $u_{0}\left(t\right)$, $u_{2k}\left(t\right)$, and $u_{2k-1}\left(t\right)$, respectively described by (3.13), (3.14), and (3.14), into (3.6), we have:

$$
\begin{aligned}
u(x,t) &= 2\varphi_0 + \frac{2}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} r(s)\, f_0(s)\, \frac{ds}{s^{1-\rho}} \\
&+ \sum_{k=1}^{+\infty} \left\{ \varphi_{2k} e_\alpha \left(\frac{t^\rho}{\rho}, \lambda_k \right) + \int_0^t e_{\alpha,\alpha} \left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k \right) r(s)\, f_{2k}(s)\, \frac{ds}{s^{1-\rho}} \right\} X_{2k}(x) \\
&+ \sum_{k=1}^{+\infty} \left\{ \varphi_{2k-1} e_\alpha \left(\frac{t^\rho}{\rho}, \lambda_k \right) + \int_0^t e_{\alpha,\alpha} \left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k \right) r(s)\, f_{2k-1}(s)\, \frac{ds}{s^{1-\rho}} \right. \\
&\left. -4\pi a k \int_0^t e_{\alpha,\alpha} \left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k \right) u_{2k}(s)\, \frac{ds}{s^{1-\rho}} \right\} X_{2k-1}(x),
\end{aligned}
\tag{3.16}
$$

Taking the generalized Caputo fractional derivative ${}^{c}\mathcal{D}_t^{\alpha,\rho}$ of the over-determination condition (3.4), and integrating the equation (3.1) on $[0,1]$ and using (3.3), we obtain

$$
r(t) = \frac{{}^{c}\mathcal{D}_t^{\alpha,\rho} g(t) + (1-\beta)\, u_x(0,t)}{\int_0^1 f(x,t)\, dx} \text{ where } \int_0^1 f(x,t)\, dx = 2 f_0(t) + \frac{2a}{\pi} \sum_{k=1}^{+\infty} \frac{f_{2k}(t)}{k},
$$

and

$$
u_x(0,t) = \sum_{k=1}^{+\infty} 8\pi k (1-b) \left(\varphi_{2k} e_\alpha \left(\frac{t^\rho}{\rho}, \lambda_k \right) + \int_0^t e_{\alpha,\alpha} \left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k \right) r(s)\, f_{2k}(s)\, \frac{ds}{s^{1-\rho}} \right).
$$

Hence, we get following implicit representation of $r(t)$

$$
r(t) = \eta(t) + \left[2 f_0(t) + \frac{2a}{\pi} \sum_{k=1}^{+\infty} \frac{f_{2k}(t)}{k} \right]^{-1} \int_0^t K(t,s)\, r(s)\, \frac{ds}{s^{1-\rho}}, \tag{3.17}
$$

where

$$
\eta(t) = \frac{{}^{c}\mathcal{D}_t^{\alpha,\rho} g(t) + a \sum_{k=1}^{+\infty} 8\pi k \varphi_{2k} e_\alpha \left(\frac{t^\rho}{\rho}, \lambda_k \right)}{\int_0^1 f(x,t)\, dx}, \tag{3.18}
$$

and

$$
K(t,s) = a \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \sum_{k=1}^{+\infty} 8\pi k f_{2k}(s)\, E_{\alpha,\alpha} \left[-\lambda_k \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha} \right]. \tag{3.19}
$$

Step 2: Existence of the solution. We consider the following map:

$$\mathcal{P}\left(r\left(t\right)\right):=\eta\left(t\right)+\left[2f_{0}\left(t\right)+\frac{2a}{\pi}\sum_{k=1}^{+\infty}\frac{f_{2k}\left(t\right)}{k}\right]^{-1}\int_{0}^{t}K\left(t,s\right)r\left(s\right)\frac{ds}{s^{1-\rho}}.$$

on the space $\mathcal{C}\left[0,T\right]$ with $\|\phi\|:=\max_{0\leq t\leq T}\left|\phi\left(t\right)\right|$. To show $\mathcal{P}$ is well defined. Since, under the assumptions (A1), (A2) and integration by parts four times, for $t,s\in\left[0,T\right]$, we obtain

$$\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)\leq\sum_{k=1}^{+\infty}\frac{L_{1}\left|\varphi_{2k}^{(4)}\right|}{2\pi^{3}k^{3}},\tag{3.20}$$

$$\sum_{k=1}^{+\infty}8\pi kf_{2k}\left(s\right)E_{\alpha,\alpha}\left[-\lambda_{k}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right]\leq\sum_{k=1}^{+\infty}\frac{L_{2}\left|f_{2k}^{(4)}\left(s\right)\right|}{2\pi^{3}k^{3}},\tag{3.21}$$

where $\varphi_{2k}^{(4)}=\int_{0}^{1}\varphi^{(4)}\left(x\right)\sin\left(2\pi kx\right)dx$ and $f_{2k}^{(4)}\left(t\right)=\int_{0}^{1}\frac{\partial^{4}f(x,t)}{\partial x^{4}}\sin\left(2\pi kx\right)dx$.

Using the Cauchy-Schwarz and Bessel inequalities, we obtain

$$\sum_{k=1}^{+\infty}\frac{L_{2}\left|f_{2k}^{(4)}\left(s\right)\right|}{2\pi^{3}k^{3}}\leq\left[\sum_{k=1}^{+\infty}\frac{L_{2}^{2}}{4\pi^{6}k^{6}}\right]^{1/2}\left[\sum_{k=1}^{+\infty}\left(f_{2k}^{(4)}\left(s\right)\right)^{2}\right]^{1/2}\leq c\left\|\frac{\partial^{4}f\left(x,t\right)}{\partial x^{4}}\right\|_{L^{2}(0,1)}$$

where c is a constant independent of t and k. Thus, we have

$$\sum_{k=1}^{+\infty}8\pi kf_{2k}\left(s\right)E_{\alpha,\alpha}\left[-\lambda_{k}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right]\leq C',\ C'=c\max_{0\leq t\leq T}\left\|\frac{\partial^{4}f\left(x,t\right)}{\partial x^{4}}\right\|_{L^{2}(0,1)}.\tag{3.22}$$

By (3.20) and (3.21), the series functions

$$\sum_{k=1}^{+\infty}8\pi kf_{2k}\left(s\right)E_{\alpha,\alpha}\left[-\lambda_{k}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right]\text{ and }\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)$$

are uniformly convergent. Then, $\eta\left(t\right)$ and $K\left(t,s\right)$ are continuous functions on $\left[0,T\right]$ and $\left[0,T\right]\times\left[0,T\right]$, respectively. Hence, the operator $\mathcal{P}$ is well defined.

Let $r_{1},r_{2}\in\mathcal{C}\left(0,T\right)$. From (3.22) and the change of variable $\tau=\frac{t^{\rho}-s^{\rho}}{\rho}$, we get

$$\left\|\mathcal{P}\left(r_{1}\right)-\mathcal{P}\left(r_{2}\right)\right\|\leq\frac{MC'\left|1-\beta\right|T^{\rho\alpha}}{\alpha\left|1+\beta\right|\rho^{\alpha}}\left\|r_{1}-r_{2}\right\|.\tag{3.23}$$

With the condition (3.5), $\frac{MC'|1-\beta|T^{\rho\alpha}}{\alpha|1+\beta|\rho^{\alpha}}<1$, then the mapping $\mathcal{P}$ is a contraction. Consequently, by Banach fixed point theorem, the mapping $\mathcal{P}$ has a unique fixed point $r \in \mathcal{C}\,[0,T]$.

To establish the regularity of the obtained solution, it remains to show

$$u\,(x,t)\,, u_x\,(x,t)\,, u_{xx}\,(x,t)\,, {}^{c}\mathcal{D}_t^{\alpha,\rho} u\,(x,t) \in \mathcal{C}\,(\Omega_T)\,.$$

Under assumptions (A1)-(A2) and integration by parts four times, we have

$$\begin{aligned} f_{2k}\,(t) &= \frac{f_{2k}^{(4)}\,(t)}{16\pi^4 k^4},\ f_{2k-1}\,(t) = \frac{-1}{16\pi^4 k^4}\left(f_{2k-1}^{(4)}\,(t) + \frac{a}{\pi k} f_{2k}^{(4)}\,(t)\right), \\ \varphi_{2k} &= \frac{\varphi_{2k}^{(4)}}{16\pi^4 k^4},\ \varphi_{2k-1} = \frac{-1}{16\pi^4 k^4}\left(\varphi_{2k-1}^{(4)} + \frac{a}{\pi k}\varphi_{2k}^{(4)}\right). \end{aligned} \tag{3.24}$$

From (3.13)-(3.15), (3.24) and (1.1), we get

$$\begin{aligned} &|u_0\,(t)| \le M_2 + \frac{M_0 M_1 T^{\rho\alpha}}{\rho^{\alpha}\Gamma\,(\alpha+1)} := M_3,\ t \in [0,T]\,, \\ &|u_{2k}\,(t)| \le \frac{L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}}{16\pi^4 k^4},\ t \in [0,T]\,, \\ &|u_{2k-1}\,(t)| \le \frac{\left(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}\right)\left(1+|a|+|a|\,CT^{\rho}/\rho\right)}{16\pi^4 k^4},\ t \in [\epsilon,T]\,,\ \epsilon > 0. \end{aligned} \tag{3.25}$$

By using (3.6) and (3.25), following relations hold for $x \in [0,1]$ and $t \in [\epsilon,T]$ with $\epsilon > 0$ such that

$$\begin{aligned} |u\,(x,t)| \le{}& 2M_1 + \sum_{k=1}^{+\infty} \frac{\left(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}\right)\left(1+|a|+|a|\,CT^{\rho}/\rho\right)}{4\pi^4 k^4} \\ &+ \sum_{k=1}^{+\infty} \frac{\left(1+|b|+|a|\right)\left(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}\right)}{4\pi^4 k^4}, \\ |u_x\,(x,t)| \le{}& \sum_{k=1}^{+\infty} \frac{\left(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}\right)\left(1+|a|+|a|\,CT^{\rho}/\rho\right)}{2\pi^3 k^3} \\ &+ \sum_{k=1}^{+\infty} \frac{\left(|a| + 2\pi k\left(1+|b|+|a|\right)\right)\left(1+|b|+|a|\right)\left(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}\right)}{4\pi^4 k^4} \\ |u_{xx}\,(x,t)| \le{}& \sum_{k=1}^{+\infty} \frac{\left(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}\right)\left(1+|a|+|a|\,CT^{\rho}/\rho\right)}{\pi k^2} \\ &+ \sum_{k=1}^{+\infty} \frac{\left(a + \pi k\left(1+|b|+|a|\right)\right)\left(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^{\alpha}\right)}{\pi^3 k^3}. \end{aligned} \tag{3.26}$$

From (3.10)-(3.12), (3.25) and for $t \in [\epsilon, T]$, we have

$$\begin{aligned}
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_0(t)\right| &\leq M_0 M_2, \\
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_{2k}(t)\right| &\leq \frac{M_0 M_2}{16\pi^4 k^4} + \frac{L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha}{4\pi^2 k^2}, \\
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_{2k-1}(t)\right| &\leq \frac{(1+|a|) M_0 M_2}{16\pi^4 k^4} + \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a| C T^\rho/\rho)}{4\pi^2 k^2} \\
&\quad + \frac{|a|(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{4\pi^3 k^3}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u(x,t)\right| \leq 2M_0 M_2 &+ \sum_{k=1}^{+\infty} \frac{(2+|b|+2|a|) M_0 M_2}{4\pi^4 k^4} \\
&+ \sum_{k=1}^{+\infty} \frac{|a|(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{\pi^3 k^3} \\
&+ \sum_{k=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(2+|b|+2|a|+|a| C T^\rho/\rho)}{\pi^2 k^2}.
\end{aligned} \tag{3.27}$$

From (3.26), (3.27) and by Weierstrass M-test, the series corresponding to $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, ${}_0^c\mathcal{D}_t^{\alpha,\rho} u(x,t)$ are uniformly convergent on $[0,1] \times [\epsilon, T]$ for $\epsilon > 0$. Hence, $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, ${}_0^c\mathcal{D}_t^{\alpha,\rho} u(x,t)$ are continuous functions on Ω_T.

Step 3: Uniqueness of the solution . Let $\{u(x,t), r_1(t)\}$ and $\{v(x,t), r_2(t)\}$ be two solution sets of the inverse problem (3.1)-(3.4). By using (3.6), we obtain

$$\begin{aligned}
u(x,t) - v(x,t) = 2(u_0(t) - v_0(t)) &+ \sum_{k=1}^{+\infty} (u_{2k-1}(t) - v_{2k-1}(t)) X_{2k-1}(x) \\
&+ \sum_{k=1}^{+\infty} (u_{2k}(t) - v_{2k}(t)) X_{2k}(x),
\end{aligned} \tag{3.28}$$

Due to the estimate (3.23) and condition (3.5), we have $r_1 = r_2$, and by substituting $r_1 = r_2$ in (3.28) and (3.13)-(3.15), we obtain $u = v$. □

3.2.2 Continuous dependence of the solution on the data

Let $\mathcal{H}$ be the set of triples $\{\varphi, f, g\}$ where the functions φ, f and g satisfy the assumptions of Theorem 3.1 and

$$\|\varphi\|_{\mathcal{C}^4(0,1)} \leq M_4,\ \|f\|_{\mathcal{C}^4(\Omega_T)} \leq M_5,\ \|g\|_{\mathcal{C}^1(0,1)} \leq M_6.$$

For $\phi \in \mathcal{H}$, we define the norm

$$\|\phi\|_{\mathcal{H}} := \|\varphi\|_{\mathcal{C}^4(0,1)} + \|f\|_{\mathcal{C}^4(\Omega_T)} + \|g\|_{\mathcal{C}^1(0,1)}\,.$$

By using the Cauchy-Schwarz and Bessel inequalities, the series functions

$$\sum_{k=1}^{+\infty} \frac{\left| f_{2k}^{(4)}\left(s\right)\right|}{2\pi^3 k^3} \leq M_7,$$

is uniformly convergent, where $f_{2k}^{(4)}\left(s\right)$ are the coefficients of the sine Fourier expansion of the function $\frac{\partial^4 f(x,s)}{\partial x^4}$.

Theorem 3.2. *The solution $\{u\left(x,t\right), r\left(t\right)\}$ of the inverse problem (3.1)-(3.4) under the assumptions of Theorem 3.1, depends continuously upon the data for $T < \left(\frac{\alpha|1+\beta|\rho^{\alpha}}{MC'|1-\beta|}\right)^{1/\rho\alpha}$.*

Proof. Let $\{u\left(x,t\right), r\left(t\right)\}$ and $\{\tilde{u}\left(x,t\right), \tilde{r}\left(t\right)\}$ be two solution sets of the inverse problem (3.1)-(3.4), corresponding to the data $\phi = \{\varphi, f, g\}$ and $\phi = \left\{\tilde{\varphi}, \tilde{f}, \tilde{g}\right\}$, respectively.

For $g, \tilde{g} \in C^1\left(0,T\right)$, we have

$$\|{}_0^c\mathcal{D}_t^{\alpha,\rho} g - {}_0^c\,\mathcal{D}_t^{\alpha,\rho}\tilde{g}\|_{\mathcal{C}(0,T)} \leq M_8 \left\|g - \tilde{g}\right\|_{\mathcal{C}^1(0,T)}\,,$$

where $M_8 = \frac{T^{1-\rho\alpha}}{\rho^{1-\alpha}\Gamma(2-\alpha)}$. From (3.18), we have

$$\eta(t) - \tilde{\eta}(t) = \left(\int_0^1 f(x,t)\,dx \int_0^1 \tilde{f}(x,t)\,dx\right)^{-1} \left[\int_0^1 \tilde{f}(x,t)\,dx \left({}_0^c\mathcal{D}_t^{\alpha,\rho} g(t) - {}_0^c\mathcal{D}_t^{\alpha,\rho}\tilde{g}(t)\right)\right.$$
$$+a\sum_{k=1}^{+\infty} 8\pi k\left(\varphi_{2k} - \tilde{\varphi}_{2k}\right) E_\alpha\left(-\lambda_k\left(\frac{t^\rho}{\rho}\right)^\alpha\right)$$
$$+{}_0^c\mathcal{D}_t^{\alpha,\rho}\tilde{g}(t)\left(\int_0^1 \tilde{f}(x,t)\,dx - \int_0^1 f(x,t)\,dx\right)$$
$$\left.+a\sum_{k=1}^{+\infty} 8\pi k\tilde{\varphi}_{2k} E_\alpha\left(-\lambda_k\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\left(\int_0^1 \tilde{f}(x,t)\,dx - \int_0^1 f(x,t)\,dx\right)\right].$$

From (3.24), we have

$$\varphi_{2k} - \tilde{\varphi}_{2k} = \int_0^1 \left(\varphi(x) - \tilde{\varphi}(x)\right) X_{2k}(x)\,dx = \frac{\varphi_{2k}^{(4)} - \tilde{\varphi}_{2k}^{(4)}}{16\pi^4 k^4}.$$

We have the estimate

$$\|\eta - \tilde{\eta}\| \le N_1 \|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4(0,1)} + N_2 \left\|f - \tilde{f}\right\|_{\mathcal{C}(\Omega_T)} + N_3 \|g - \tilde{g}\|_{C^1(0,1)},$$

where $N_1 = M^2 |a| L_1 C^*$, $N_2 = M^2 (|a| L_1 M_7 + M_6 M_8)$, $N_3 = M^2 M_5 M_8$.

From (3.17), we have the estimate

$$\|r - \tilde{r}\| \le \|\eta - \tilde{\eta}\| + \frac{M M_0 |a| T^{\rho\alpha}}{\alpha\rho^\alpha}\left\|f^{(4)} - \tilde{f}^{(4)}\right\|_{\mathcal{C}(\Omega_T)} + \frac{M |a| C' T^{\rho\alpha}}{\alpha\rho^\alpha}\|r - \tilde{r}\|$$
$$+ \frac{M^2 M_0 C' T^{\rho\alpha}}{\alpha\rho^\alpha}\left\|f - \tilde{f}\right\|_{\mathcal{C}(\Omega_T)}.$$

Due to the estimate of $\|\eta - \tilde{\eta}\|$, we have

$$\left(1 - \frac{M |a| C' T^{\rho\alpha}}{\alpha\rho^\alpha}\right)\|r - \tilde{r}\| \le N_1 \|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4(0,1)}$$
$$+ \left(N_2 + \frac{M M_0 |a| T^{\rho\alpha}}{\alpha\rho^\alpha} + \frac{M^2 M_0 C' T^{\rho\alpha}}{\alpha\rho^\alpha}\right)\left\|f - \tilde{f}\right\|_{\mathcal{C}^4(\Omega_T)}$$
$$+ N_3 \|g - \tilde{g}\|_{C^1(0,1)}.$$

Hence

$$\left(1 - \frac{M\,|a|\,C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}\right)\|r - \tilde{r}\| \leq N_4 \left\|\phi - \tilde{\phi}\right\|_{\mathcal{H}},$$

where $N_4 := \max\left\{N_1, N_2 + \frac{MM_0|a|T^{\rho\alpha}}{\alpha\rho^{\alpha}} + \frac{M^2M_0C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}, N_3\right\}$. For $T < \left(\frac{\alpha\rho^{\alpha}}{M|a|C'}\right)^{1/\rho\alpha}$, we have

$$\|r - \tilde{r}\| \leq \frac{N_4}{1 - \frac{M|a|C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}} \left\|\phi - \tilde{\phi}\right\|_{\mathcal{H}}.$$

From (3.6), a similar estimate can be also obtained for the difference $u - \tilde{u}$:

$$\|u - \tilde{u}\|_{\mathcal{C}(\bar{\Omega}_T)} \leq N_5 \left\|\phi - \tilde{\phi}\right\|_{\mathcal{H}}.$$

□

Chapter 4

An inverse time-dependent coefficient problem

In this chapter, we study an inverse problem of determining the time-dependent coefficient in one-dimensional time-fractional reaction-diffusion equation with nonlocal boundary and overdeterminarion conditions. The time-fractional derivative is described in the conformable sense. Under some assumptions on the input data, the well-posedness of this inverse time-dependent coefficient problem is shown by using Fourier's method and Banach's contraction mapping principle.

4.1 Statement of the problem

In this section, we consider the time-fractional reaction-diffusion equation

$$\mathcal{D}_t^{(\alpha)} w(x,t) = w_{xx}(x,t) - p(t)\, w(x,t) + S(x,t)\,,\ (x,t) \in \Omega_T, \tag{4.1}$$

where $\mathcal{D}_t^{(\alpha)}$ represent the left-conformable fractional derivative of order $0 < \alpha \leq 1$ with respect to t, $S(x,t)$ is the source term and $w(x,t)$ represent the temperature in a segment slab $[0,1]$ over time interval $]\,0, T\,]$ with $T > 0$, $p(t)$ describes the coefficient of heat capacity.

For $\alpha = 1$, equation (4.1) is a classical reaction-diffusion equation. Suppose the unknown function w satisfy the following initial condition

$$w(x,0) = \varphi(x)\,,\ 0 \leq x \leq 1, \tag{4.2}$$

and nonlocal family of boundary conditions

$$\begin{cases} w(0,t) = w(1,t), \\ \beta w_x(0,t) = w_x(1,t), \end{cases} \qquad 0 < t \le T, \tag{4.3}$$

where φ are given function and β is a real number such that $-1 < \beta < 1$.

When the coefficient $p(t)$ is given, the problem of finding $w(x,t)$ from the equation (4.1), initial condition (4.2) and boundary conditions (4.3) is referred to as the direct problem. In the case when the function $p(t)$ is unknown, the inverse problem we are interested in consists in determining a pair of functions $\{w(x,t), p(t)\}$ satisfying equation (4.1), initial condition (4.2), boundary conditions (4.3) and overdetermination condition

$$\int_0^1 w(x,t)\,dx = E(t),\ 0 \le t \le T, \tag{4.4}$$

where $E(t)$ is a given function. We begin our investigation with a pair of transformations:

$$v(x,t) = \mu(t)\,w(x,t),\ \mu(t) = \exp\left(\int_0^t s^{\alpha-1} p(s)\,ds\right),\ 0 < \alpha \le 1,\ t \in [0,T]. \tag{4.5}$$

Then, the inverse time-dependent coefficient problem given by (4.1)-(4.4) transforms as

$$\mathcal{D}_t^{(\alpha)} v(x,t) = v_{xx}(x,t) + \mu(t)\,S(x,t),\ (x,t) \in \Omega_T, \tag{4.6}$$

$$v(x,0) = \varphi(x),\ x \in [0,1], \tag{4.7}$$

$$v(0,t) = v(1,t),\ \beta v_x(0,t) = v_x(1,t),\ t \in [0,T], \tag{4.8}$$

$$\int_0^1 v(x,t)\,dx = \mu(t)\,E(t),\ t \in [0,T], \tag{4.9}$$

where $\mu(0) = 1$ and $\mu(t) > 0$ for $t \in [0,T]$. Solving the inverse time-dependent source problem given by (4.6)-(4.9) for the solution pair $(v(x,t), \mu(t))$ yields afterwards the original solution $(w(x,t), p(t))$ for the inverse time-dependent coefficient problem given by (4.1)-(4.4) from

$$w(x,t) = \frac{v(x,t)}{\mu(t)},\ p(t) = \frac{\mathcal{D}_t^{(\alpha)}\mu(t)}{\mu(t)}. \tag{4.10}$$

4.2 Well-posedness of the problem

4.2.1 Existence and uniqueness of the solution

In this subsection, we present the main result on the existence and uniqueness of the solution to the inverse time-dependent source problem (4.6)-(4.9).

Theorem 4.1. *Suppose that the following assumptions hold:*

(A_1)**:** $\varphi \in \mathcal{C}^4[0,1]$; $\varphi(0) = \varphi(1)$; $\varphi'(1) = \beta\varphi'(0)$; $\varphi''(0) = \varphi''(1)$; $\varphi'''(1) = \beta\varphi'''(0)$,

(A_2)**:** $S(\cdot,t) \in \mathcal{C}^4[0,1]$, *for all* $t \in [0,T]$; $S(0,t) = S(1,t)$; $S_x(1,t) = \beta S_x(0,t)$; $S_{xx}(0,t) = S_{xx}(1,t)$; $S_{xxx}(1,t) = \beta S_{xxx}(0,t)$,

(A_3)**:** *E is $\alpha-$differentiable and* $\mathcal{D}^{(\alpha)}E \in \mathcal{C}[0,T]$, $E(t) \neq 0$, $\int_0^1 S(x,\cdot)\,dx \in \mathcal{C}[0,T]$, *for all* $t \in [0,T]$.

If the following condition hold:

$$T < \left(\frac{\alpha m}{M + \frac{\omega(1-\beta)}{8\pi^5(1+\beta)}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}}\right)^{1/\alpha}, \tag{4.11}$$

where $m = \min\limits_{0\le t\le T}|E(t)|$, $M = \max\limits_{0\le t\le T}\left|\int_0^1 S(x,t)\,dx - \mathcal{D}^{(\alpha)}E(t)\right|$, $\omega = \sum\limits_{k=1}^{+\infty}\frac{1}{k^5}$ *and*

$$S_{2k}^{(4)}(t) = \int_0^1 S_{xxxx}(x,t)\sin(2\pi kx)\,dx.$$

Then, the inverse time-dependent source problem (4.6)-(4.9) has a unique solution $\{v(x,t),\mu(t)\}$.

Proof. According to assumptions $(A_1)-(A_3)$, there are positive constants

$$M_0 := \|\mu\|_{\mathcal{C}[0,T]},\ M_1 := \max\left(\|S_0\|_{\mathcal{C}[0,T]}, \left\|S_{2k-1}^{(4)}\right\|_{\mathcal{C}[0,T]}, \left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}\right),\ M_2 := \max\left(|\varphi_0|, \left|\varphi_{2k-1}^{(4)}\right|, \left|\varphi_{2k}^{(4)}\right|\right)$$

The proof of this theorem takes place in three steps:

Step 1: Construction of solution . By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of the direct problem (4.6)-(4.8) for μ is $\alpha-$differentiable on $[0,T]$,

$$v(x,t)=2v_0(t)+\sum_{k=1}^{+\infty}v_{2k-1}(t)X_{2k-1}(x)+\sum_{k=1}^{+\infty}v_{2k}(t)X_{2k}(x). \tag{4.12}$$

We define the coefficients $v_0(t)$, $v_{2k-1}(t)$ and $v_{2k}(t)$ for $k\in\mathbb{N}^*$ by multiplying (4.12) by eign-and associated functions of (2.46) and integrating over $[0,1]$ and using Lemma 2.2, we get

$$v_0(t)=\langle v(x,t),Y_0(x)\rangle,\ v_{2k-1}(t)=\langle v(x,t),Y_{2k-1}(x)\rangle,\ v_{2k}(t)=\langle v(x,t),Y_{2k}(x)\rangle, \tag{4.13}$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in $L^2(0,1)$.

The expansion coefficients of the functions $S(x,t)$ and $\varphi(x)$ into eigen-and associated functions (2.46) are given by

$$S_0(t)=\langle S(x,t),Y_0(x)\rangle,\ S_{2k-1}(t)=\langle S(x,t),Y_{2k-1}(x)\rangle,\ S_{2k}(t)=\langle S(x,t),Y_{2k}(x)\rangle, \tag{4.14}$$

and

$$\varphi_0=\langle\varphi(x),Y_0(x)\rangle,\ \varphi_{2k-1}=\langle\varphi(x),Y_{2k-1}(x)\rangle,\ \varphi_{2k}=\langle\varphi(x),Y_{2k}(x)\rangle. \tag{4.15}$$

From (4.6), (4.12)-(4.15), Lemma 2.2, integration by parts twice and (4.8), we obtain

$$\begin{cases}\mathcal{D}_t^{(\alpha)}v_0(t)=\mu(t)S_0(t),\\ v_0(0)=\varphi_0,\end{cases} \tag{4.16}$$

$$\begin{cases}\mathcal{D}_t^{(\alpha)}v_{2k-1}(t)+\lambda_k v_{2k-1}(t)=-4\pi k a v_{2k}(t)+\mu(t)S_{2k-1}(t),\\ v_{2k-1}(0)=\varphi_{2k-1},\end{cases} \tag{4.17}$$

$$\begin{cases}\mathcal{D}_t^{(\alpha)}v_{2k}(t)+\lambda_k v_{2k}(t)=\mu(t)S_{2k}(t),\\ v_{2k}(0)=\varphi_{2k},\end{cases} \tag{4.18}$$

Applying $\mathcal{I}_\alpha$ on (4.16) and using Lemma 1.2, we obtain

$$v_0(t) = \varphi_0 + \int_0^t \mu(s) S_0(s) \frac{ds}{s^{1-\alpha}}. \tag{4.19}$$

Applying Theorem 1.8 on (4.17) and (4.18) , we obtain:

$$\begin{aligned} v_{2k-1}(t) = \varphi_{2k-1} \exp\left(-\lambda_k \frac{t^\alpha}{\alpha}\right) - 4\pi k a \int_0^t \exp\left(\lambda_k \frac{s^\alpha - t^\alpha}{\alpha}\right) v_{2k}(s) \frac{ds}{s^{1-\alpha}} \\ + \int_0^t \exp\left(\lambda_k \frac{s^\alpha - t^\alpha}{\alpha}\right) \mu(s) S_{2k-1}(s) \frac{ds}{s^{1-\alpha}}, \end{aligned} \tag{4.20}$$

and

$$v_{2k}(t) = \varphi_{2k} \exp\left(-\lambda_k \frac{t^\alpha}{\alpha}\right) + \int_0^t \exp\left(\lambda_k \frac{s^\alpha - t^\alpha}{\alpha}\right) \mu(s) S_{2k}(s) \frac{ds}{s^{1-\alpha}}. \tag{4.21}$$

After substituting expressions $v_0(t)$, $v_{2k-1}(t)$, and $v_{2k}(t)$, respectively described by (4.19), (4.20), and (4.21), into (4.12), we have:

$$\begin{aligned} v(x,t) =& 2\varphi_0 + 2\int_0^t \mu(s) S_0(s) \frac{ds}{s^{1-\alpha}} \\ & \sum_{k=1}^{+\infty} \left[\varphi_{2k-1} \exp\left(-\lambda_k \frac{t^\alpha}{\alpha}\right) - 4\pi k a \int_0^t \exp\left(\lambda_k \frac{s^\alpha - t^\alpha}{\alpha}\right) v_{2k}(s) \frac{ds}{s^{1-\alpha}} \right. \\ & \left. + \int_0^t \exp\left(\lambda_k \frac{s^\alpha - t^\alpha}{\alpha}\right) \mu(s) S_{2k-1}(s) \frac{ds}{s^{1-\alpha}} \right] X_{2k-1}(x) \\ & \sum_{k=1}^{+\infty} \left[\varphi_{2k} \exp\left(-\lambda_k \frac{t^\alpha}{\alpha}\right) + \int_0^t \exp\left(\lambda_k \frac{s^\alpha - t^\alpha}{\alpha}\right) \mu(s) S_{2k}(s) \frac{ds}{s^{1-\alpha}} \right] X_{2k}(x). \end{aligned} \tag{4.22}$$

Taking the conformable fractional derivative $\mathcal{D}_t^{(\alpha)}$ of the over-determination condition (4.9), and integrating the equation (4.6) on $[0,1]$ and using (4.8) and (4.22), we obtain the Cauchy problem involving a fractional integro-differential equation given by:

$$\begin{cases} \mathcal{D}_t^{(\alpha)} \mu(t) = \eta(t) + \delta(t) \mu(t) + \int_0^t K(t,s) \mu(s) \, ds, \\ \mu(0) = 1, \end{cases} \tag{4.23}$$

where

$$\eta(t) = -\frac{8\pi a}{E(t)} \sum_{k=1}^{+\infty} k\varphi_{2k} \exp\left(-\lambda_k \frac{t^\alpha}{\alpha}\right), \tag{4.24}$$

$$\delta(t) = \frac{\int_0^1 S(x,t)\,dx - \mathcal{D}_t^{(\alpha)} E(t)}{E(t)}, \tag{4.25}$$

$$K(t,s) = -\frac{8\pi a}{E(t)} \sum_{k=1}^{+\infty} kS_{2k}(s) \exp\left(\lambda_k \frac{s^\alpha - t^\alpha}{\alpha}\right) s^{\alpha-1}. \tag{4.26}$$

Using Lemma 1.2, the Cauchy problem (4.23) is equivalent to the following linear integral equation

$$\mu(t) = 1 + \int_0^t s^{\alpha-1}\left[\eta(s) + \delta(s)\mu(s) + \int_0^s K(s,\tau)\mu(\tau)\,d\tau\right] ds. \tag{4.27}$$

Step 2: Existence of the solution. From (4.27), we define on the space $\mathcal{C}[0,T]$ the following operator:

$$\Phi(\mu(t)) := 1 + \int_0^t s^{\alpha-1}\left[\eta(s) + \delta(s)\mu(s) + \int_0^s K(s,\tau)\mu(\tau)\,d\tau\right] ds. \tag{4.28}$$

To show Φ is well defined. Under the assumptions (A_1) and (A_2), and from (4.24), (4.26) and using integration by parts, we have:

$$\int_0^t s^{\alpha-1}\eta(s)\,ds = \sum_{k=1}^{+\infty} \int_0^t \frac{8\pi k a\varphi_{2k}}{E(s)} s^{\alpha-1} \exp\left(-\lambda_k \frac{s^\alpha}{\alpha}\right) ds \leq \sum_{k=1}^{+\infty} \frac{a}{8\pi^5 m} \frac{\left|\varphi_{2k}^{(4)}\right|}{k^5}, \tag{4.29}$$

and

$$\begin{aligned}\int_0^t s^{\alpha-1}\left[\int_0^s K(s,\tau)\mu(\tau)\,d\tau\right] ds &= \sum_{k=1}^{+\infty} \int_0^t s^{\alpha-1}\left[\int_0^s \frac{8\pi k a S_{2k}(\tau)\mu(\tau)}{E(s)} \tau^{\alpha-1} \exp\left(\lambda_k \frac{\tau^\alpha - s^\alpha}{\alpha}\right) d\tau\right] ds \\ &\leq \sum_{k=1}^{+\infty} \frac{aT^\alpha \|\mu\|_{\mathcal{C}[0,T]} \left|S_{2k}^{(4)}(\tau)\right|}{8m\alpha\pi^5 k^5},\end{aligned} \tag{4.30}$$

where $\varphi_{2k}^{(4)} = \int_0^1 \varphi^{(4)}(x)\sin(2\pi kx)\,dx$ and $S_{2k}^{(4)}(s) = \int_0^1 \frac{\partial^4 S}{\partial x^4}(x,s)\sin(2\pi kx)\,dx$. From (4.29) and (4.30), the series functions

$$\sum_{k=1}^{+\infty} \int_0^t \frac{8\pi k a\varphi_{2k}}{E(s)} s^{\alpha-1} \exp\left(-\lambda_k \frac{s^\alpha}{\alpha}\right) ds$$

and

$$\sum_{k=1}^{+\infty}\int_0^t s^{\alpha-1}\left[\int_0^s \frac{8\pi k a S_{2k}(\tau)\mu(\tau)}{E(s)}\tau^{\alpha-1}\exp\left(\lambda_k\frac{\tau^\alpha - s^\alpha}{\alpha}\right)d\tau\right]ds$$

are uniformly convergent. Then, $t\mapsto \int_0^t s^{\alpha-1}\eta(s)\,ds$ and $t\mapsto \int_0^t s^{\alpha-1}\left[\int_0^s K(s,\tau)\mu(\tau)\,d\tau\right]ds$ are continuous functions on $[0,T]$, respectively.

Under assumption (A_3) and using (4.25), we obtain $t\mapsto \int_0^t s^{\alpha-1}\delta(s)\mu(s)\,ds$ is continuous function. Hence, the operator Φ is well defined. Now we prove that Φ is a contraction operator in the space $\mathcal{C}[0,T]$. Let $\mu_1,\mu_2\in\mathcal{C}[0,T]$, using (4.28) and (4.30) we obtain

$$\begin{aligned}
|\Phi(\mu_1(t))-\Phi(\mu_2(t))| &\le \int_0^t s^{\alpha-1}\,|\delta(s)\,|\mu_1(s)-\mu_2(s)||\,ds\\
&+\int_0^t s^{\alpha-1}\left[\int_0^s |K(s,\tau)|\,|\mu_1(\tau)-\mu_2(\tau)|\,d\tau\right]ds\\
&\le \frac{T^\alpha}{\alpha m}\left[M+\frac{a\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}}{8\pi^5}\sum_{k=1}^{+\infty}\frac{1}{k^5}\right]\|\mu_1-\mu_2\|_{\mathcal{C}[0,T]}\\
&\le \frac{T^\alpha}{\alpha m}\left[M+\frac{a\omega}{8\pi^5}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}\right]\|\mu_1-\mu_2\|_{\mathcal{C}[0,T]},
\end{aligned}$$

where $\sum_{k=1}^{+\infty}\frac{1}{k^5}=\omega$. Hence,

$$\|\Phi(\mu_1)-\Phi(\mu_2)\|_{\mathcal{C}[0,T]}\le \frac{T^\alpha}{\alpha m}\left[M+\frac{\omega(1-\beta)}{8\pi^5(1+\beta)}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}\right]\|\mu_1-\mu_2\|_{\mathcal{C}[0,T]}. \tag{4.31}$$

With the condition (4.11), $\frac{T^\alpha}{\alpha m}\left[M+\frac{\omega(1-\beta)}{8\pi^5(1-\beta)}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}\right]<1$, then the mapping Φ is a contraction. Consequently, by Banach fixed point theorem, the mapping Φ has a unique fixed point $\mu\in\mathcal{C}[0,T]$.

To establish the regularity of the obtained solution, it remains to show

$$v(x,t),\ v_x(x,t),\ v_{xx}(x,t),\ \mathcal{D}_t^{(\alpha)}v(x,t)\in\mathcal{C}(\Omega_T).$$

Under assumptions $(A_1)-(A_2)$ and integration by parts four times, we have

$$\begin{aligned}
&\varphi_{2k}=\frac{\varphi_{2k}^{(4)}}{16\pi^4k^4},\ \varphi_{2k-1}=\frac{-1}{16\pi^4k^4}\left(\varphi_{2k-1}^{(4)}+\frac{a}{\pi k}\varphi_{2k}^{(4)}+\right),\\
&S_{2k}\left(t\right)=\frac{S_{2k}^{(4)}\left(t\right)}{16\pi^4k^4},\ S_{2k-1}\left(t\right)=\frac{-1}{16\pi^4k^4}\left(S_{2k-1}^{(4)}\left(t\right)+\frac{a}{\pi k}S_{2k}^{(4)}\left(t\right)\right).
\end{aligned} \tag{4.32}$$

From (4.19)-(4.21) and (4.32) we get

$$\begin{aligned}
&\left|v_0\left(t\right)\right|\leq M_2+\frac{T^\alpha}{\alpha}M_0M_1:=M_3,\ \ t\in\left[0,T\right],\\
&\left|v_{2k-1}\left(t\right)\right|\leq\frac{M_2}{16\pi^4k^4}+\frac{aM_2}{8\pi^5k^5}+\frac{M_0M_1}{64\pi^6k^6}+\frac{3aM_0M_1}{64\pi^7k^7},\\
&\left|v_{2k}\left(t\right)\right|\leq\frac{M_2}{16\pi^4k^4}+\frac{M_0M_1}{32\pi^6k^6}.
\end{aligned} \tag{4.33}$$

By using (4.12) and (4.33), following relations hold for $x\in[0,1]$ such that

$$\begin{aligned}
&\left|v\left(x,t\right)\right|\leq 2M_3+\sum_{k=1}^{+\infty}\left[\frac{M_2\left(2-b\right)}{4\pi^4k^4}+\frac{aM_2}{2\pi^5k^5}+\frac{\left(3-2b\right)M_0M_1}{16\pi^6k^6}+\frac{3aM_0M_1}{16\pi^7k^7}\right],\\
&\left|v_x\left(x,t\right)\right|\leq\sum_{k=1}^{+\infty}\left[\frac{M_2\left(2-b\right)}{2\pi^3k^3}+\frac{5aM_2}{4\pi^4k^4}+\frac{M_0M_1\left(3-2b\right)}{8\pi^5k^5}+\frac{aM_0M_1}{2\pi^6k^6}\right],\\
&\left|v_{xx}\left(x,t\right)\right|\leq\sum_{k=1}^{+\infty}\left[\frac{M_2\left(2-b\right)}{\pi^2k^2}+\frac{3aM_2}{\pi^3k^3}+\frac{M_0M_1\left(3-2b\right)}{4\pi^4k^4}+\frac{5aM_0M_1}{4\pi^5k^5}\right].
\end{aligned} \tag{4.34}$$

From (4.16)-(4.18), (4.33) we have

$$\begin{aligned}
&\left|\mathcal{D}_t^{(\alpha)}v_0\left(t\right)\right|\leq M_0M_1,\\
&\left|\mathcal{D}_t^{(\alpha)}v_{2k-1}\left(t\right)\right|\leq\frac{M_2}{4\pi^2k^2}+\frac{3aM_2}{4\pi^3k^3}+\frac{M_0M_1}{8\pi^4k^4}+\frac{3aM_0M_1}{8\pi^5k^5},\\
&\left|\mathcal{D}_t^{(\alpha)}v_{2k}\left(t\right)\right|\leq\frac{M_2}{4\pi^2k^2}+\frac{3M_0M_1}{16\pi^4k^4}.
\end{aligned}$$

Consequently,

$$\left|\mathcal{D}_t^{(\alpha)}v\left(x,t\right)\right|\leq 2M_0M_1+\sum_{k=1}^{+\infty}\left(\frac{M_2\left(2-b\right)}{\pi^2k^2}+\frac{3aM_2}{\pi^3k^3}+\frac{M_0M_1\left(5-3b\right)}{2\pi^4k^4}+\frac{3aM_0M_1}{2\pi^5k^5}\right). \tag{4.35}$$

From (4.34), (4.35) and by Weierstrass M-test, the series corresponding to $v\left(x,t\right)$, $v_x\left(x,t\right)$, $v_{xx}\left(x,t\right)$ and $\mathcal{D}_t^{(\alpha)}v\left(x,t\right)$ are uniformly convergent on Ω_T. Hence, $v\left(x,t\right)$, $v_x\left(x,t\right)$, $v_{xx}\left(x,t\right)$ and $\mathcal{D}_t^{(\alpha)}v\left(x,t\right)$ are continuous functions on Ω_T.

Step 3: Uniqueness of the solution. Let $\{u(x,t), \mu_1(t)\}$ and $\{v(x,t), \mu_2(t)\}$ be two solutions of the inverse problem (4.6)-(4.9). By using (4.12), we obtain

$$\begin{aligned} u(x,t) - v(x,t) = 2(u_0(t) - v_0(t)) &+ \sum_{k=1}^{+\infty} (u_{2k-1}(t) - v_{2k-1}(t)) X_{2k-1}(x) \\ &+ \sum_{k=1}^{+\infty} (u_{2k}(t) - v_{2k}(t)) X_{2k}(x). \end{aligned} \tag{4.36}$$

By the estimate (4.31) and condition (4.11), we have $\mu_1 = \mu_2$. Substituting $\mu_1 = \mu_2$ into equation (4.36) and (4.13)-(4.15), it follows that $u = v$. □

4.2.2 Continuous dependence of the solution on the data

In this subsection, we give the main result on continuous dependence upon the data of the solution pair $\{v(x,t), \mu(t)\}$ of the inverse time-dependent source problem (4.6)-(4.9). Let $\mathcal{B}$ be the set of triples $\{\varphi, S, E\}$, where the functions φ, S and E satisfy the assumptions of Theorem 4.1 and

$$\tilde{M} = \max_{0 \le t \le T} \left| \int \tilde{S}(x,t)\, dx - \mathcal{D}_t^{(\alpha)} \tilde{E}(t) \right|, \quad \tilde{m} = \min_{0 \le t \le T} |E(t)|.$$

For $\psi \in \mathcal{B}$, we define the norm

$$\|\psi\|_{\mathcal{B}} := \|\varphi\|_{\mathcal{C}^4[0,1]} + \|S\|_{\mathcal{C}^4[0,1] \times \mathcal{C}[0,T]} + \|E\|_{\mathcal{C}[0,T]}. \tag{4.37}$$

Theorem 4.2. *The solution $\{v(x,t), \mu(t)\}$ of the inverse time-dependent source problem (4.6)-(4.9) under the assumptions of Theorem 4.1, depends continuously upon the data if T verified the condition (4.11).*

Proof. Let $\{v(x,t), \mu(t)\}$ and $\{\tilde{v}(x,t), \tilde{\mu}(t)\}$ be two solutions of the inverse time-dependent source problem (4.6)-(4.9), corresponding to the data $\{\varphi, S, E\}$ and $\left\{\tilde{\varphi}, \tilde{S}, \tilde{E}\right\}$, respectively.

From (4.27), we have

$$\begin{aligned} |\mu(t) - \tilde{\mu}(t)| \le \int_0^t s^{\alpha-1} [\eta(s) - \tilde{\eta}(s)]\, ds &+ \int_0^t s^{\alpha-1} \left[\delta(s)\mu(s) - \tilde{\delta}(s)\tilde{\mu}(s)\right] ds \\ &+ \int_0^t s^{\alpha-1} \left[\int_0^s \left(K(s,\tau)\mu(\tau) - \tilde{K}(s,\tau)\tilde{\mu}(\tau)\right) d\tau\right] ds. \end{aligned} \tag{4.38}$$

Using (4.24), the first integral of (4.38) becomes

$$\begin{aligned}\int_0^t s^{\alpha-1}\left[\eta(s)-\tilde{\eta}(s)\right]ds &= \sum_{k=1}^{+\infty} 8\pi k a \int_0^t s^{\alpha-1}\exp\left(-\lambda_k \frac{s^\alpha}{\alpha}\right)\left[\frac{\varphi_{2k}}{E(s)}-\frac{\tilde{\varphi}_{2k}}{\tilde{E}(s)}\right]ds \\ &= \sum_{k=1}^{+\infty} 8\pi k a \int_0^t s^{\alpha-1}\exp\left(-\lambda_k \frac{s^\alpha}{\alpha}\right)\left[\frac{\varphi_{2k}-\tilde{\varphi}_{2k}}{E(s)}-\frac{\tilde{\varphi}_{2k}\left(E(s)-\tilde{E}(s)\right)}{E(s)\tilde{E}(s)}\right]ds.\end{aligned} \tag{4.39}$$

Under assumptions $(A_1)-(A_3)$ and from (4.32), (4.39) we obtain

$$\int_0^t s^{\alpha-1}\left[\eta(s)-\tilde{\eta}(s)\right]ds \le M_4\left\|\varphi-\tilde{\varphi}\right\|_{\mathcal{C}^4[-\pi,\pi]}+M_5\left\|E-\tilde{E}\right\|_{\mathcal{C}[0,T]}, \tag{4.40}$$

where

$$M_4=\frac{a}{\pi^5 m}\sum_{k=1}^{+\infty}\frac{1}{k^5},\ M_5=\frac{a\left\|\tilde{\varphi}\right\|_{\mathcal{C}^4[0,1]}}{\pi^5 m\tilde{m}}\sum_{k=1}^{+\infty}\frac{1}{k^5}.$$

Using (4.25), we have

$$\left|\delta(s)-\tilde{\delta}(s)\right| \le M_6\left\|S-\tilde{S}\right\|_{\mathcal{C}\left(\bar{\Omega}_T\right)}+M_7\left\|E-\tilde{E}\right\|_{\mathcal{C}[0,T]}, \tag{4.41}$$

where

$$M_6=1/m,\ M_7=\max\left\{1/m,\tilde{M}/m\tilde{m}\right\}.$$

With (4.41), the second integral (4.38) becomes

$$\int_0^t s^{\alpha-1}\left[\beta(s)\mu(s)-\tilde{\beta}(s)\tilde{\mu}(s)\right]ds \le \frac{MT^\alpha}{\alpha m}\left\|\mu-\tilde{\mu}\right\|_{\mathcal{C}[0,T]}+M_8\left\|S-\tilde{S}\right\|_{\mathcal{C}\left(\bar{\Omega}_T\right)}+M_9\left\|E-\tilde{E}\right\|_{\mathcal{C}[0,T]}, \tag{4.42}$$

where

$$M_8=\frac{M_6T^\alpha}{\alpha}\left\|\tilde{\mu}\right\|_{\mathcal{C}[0,T]},\ M_9=\frac{M_7T^\alpha}{\alpha}\left\|\tilde{\mu}\right\|_{\mathcal{C}[0,T]}.$$

The third integral of (4.38) becomes

$$\int_0^t s^{\alpha-1}\left[\int_0^s \left(K(s,\tau)\mu(\tau) - \tilde{K}(s,\tau)\tilde{\mu}(\tau)\right)d\tau\right]ds = \int_0^t s^{\alpha-1}\left[\int_0^s K(s,\tau)(\mu(\tau)-\tilde{\mu}(\tau))\,d\tau\right]ds + \int_0^t s^{\alpha-1}\left[\int_0^s \tilde{\mu}(\tau)\left(K(s,\tau)-\tilde{K}(s,\tau)\right)d\tau\right]ds. \tag{4.43}$$

Using (4.26), the first integral of (4.43) becomes

$$\int_0^t s^{\alpha-1}\left[\int_0^s K(s,\tau)(\mu(\tau)-\tilde{\mu}(\tau))\,d\tau\right]ds \le \frac{a\omega T^{\alpha}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}}{8\alpha m\pi^5}\|\mu-\tilde{\mu}\|_{\mathcal{C}[0,T]}, \tag{4.44}$$

and the second integral of (4.43) becomes

$$\int_0^t s^{\alpha-1}\left[\int_0^s \tilde{\mu}(\tau)\left(K(s,\tau)-\tilde{K}(s,\tau)\right)d\tau\right]ds \le M_{10}\left\|S-\tilde{S}\right\|_{\mathcal{C}^4[0,1]\times\mathcal{C}[0,T]} + M_{11}\left\|E-\tilde{E}\right\|_{\mathcal{C}^{\alpha}[0,T]}, \tag{4.45}$$

where

$$M_{10} = \frac{T^{\alpha}}{20\alpha m},\ M_{11} = \frac{T^{\alpha}\left\|\tilde{f}\right\|_{\mathcal{C}^4[0,1]\times\mathcal{C}[0,T]}}{20\alpha m\tilde{m}}.$$

From (4.38), (4.40), (4.42) and (4.43)-(4.45), we have the estimate

$$\left[1-\frac{T^{\alpha}}{\alpha m}\left(M+\frac{a\omega}{8\pi^5}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}\right)\right]\|\mu-\tilde{\mu}\|_{\mathcal{C}[0,T]} \le M_1\|\varphi-\tilde{\varphi}\|_{\mathcal{C}^4[0,1]} + M_{12}\left\|S-\tilde{S}\right\|_{\mathcal{C}^4[0,1]\times\mathcal{C}[0,T]} + M_{13}\left\|E-\tilde{E}\right\|_{\mathcal{C}^{\alpha}[0,T]}, \tag{4.46}$$

where $M_{12} = M_8 + M_{10},\ M_{13} = M_9 + M_{11}$. From (4.11) and (4.37), we get

$$\left[1-\frac{T^{\alpha}}{\alpha m}\left(M+\frac{a\omega}{8\pi^5}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}\right)\right]\|\mu-\tilde{\mu}\|_{\mathcal{C}[0,T]} \le M_{14}\left\|\psi-\tilde{\psi}\right\|_{\mathcal{B}}, \tag{4.47}$$

where $M_{14} = \max\{M_1, M_{12}, M_{13}\}$. From (4.11) and (4.37), we have

$$\|\mu-\tilde{\mu}\|_{\mathcal{C}[0,T]} \le M_{15}\left\|\psi-\tilde{\psi}\right\|_{\mathcal{B}}, \tag{4.48}$$

where

$$M_{15} = \frac{M_{14}}{1 - \frac{T^{\alpha}}{\alpha m}\left(M + \frac{a\omega}{8\pi^5}\left\|S_{2k}^{(4)}\right\|_{\mathcal{C}[0,T]}\right)}.$$

Under assumptions $(A_1) - (A_3)$ and using (4.12), we obtain

$$\|v - \tilde{v}\|_{\mathcal{C}(\bar{\Omega}_T)} \leq M_{16}\,\|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[-\pi,\pi]} + M_{17}\,\|f - f\|_{\mathcal{C}^4[-\pi,\pi]\times\mathcal{C}[0,T]} + M_{18}\,\|\mu - \tilde{\mu}\|_{\mathcal{C}[0,T]}\,, \tag{4.49}$$

where

$$M_{16} = 1 + \frac{\pi^4}{10},\ M_{17} = \frac{T^{\alpha}\,\|\tilde{\mu}\|_{\mathcal{C}[0,T]}}{\alpha} + \frac{\|\tilde{\mu}\|_{\mathcal{C}[0,T]}}{10},\ M_{18} = \frac{T^{\alpha}\,\|S\|_{\mathcal{C}^4[0,1]\times\mathcal{C}[0,T]}}{\alpha} + \frac{\|f\|_{\mathcal{C}^4[0,1]\times\mathcal{C}[0,T]}}{10}$$

From (4.48) and (4.49), we get

$$\|v - \tilde{v}\|_{\mathcal{C}(\bar{\Omega}_T)} + \|\mu - \tilde{\mu}\|_{\mathcal{C}[0,T]} \leq M_{19}\left\|\psi - \tilde{\psi}\right\|_{\mathcal{B}},$$

where $M_{19} = \max\{M_{15}, M_{16}, M_{17}, M_{18}\}$. Then, the solution of the inverse problem (4.6)-(4.9) is depends continuously upon the data. □

Chapter 5

Finite difference approximation for the inverse time-dependent source problem

In this chapter we investigate the finit difference methods for an inverse time-dependent source problem for a time-fractional diffusion equation with nonlocal boundary and integral over determination conditions.

Let $T>0$ be a Fixed number and $D_T=\{(x,t): 0<x<1,\ 0<t\leq T\}$ we are concerned with the following fractional differential equation in $\overline{D}_T$:

$$ {}^{C}\mathcal{D}^{\alpha,\rho}_{0^+}(u(x,t)) = u_{xx}(x,t)+r(t)f(x,t), \tag{5.1} $$

with the initial condition

$$ u(x,0)=\varphi(x),\ 0<x<1, \tag{5.2} $$

and the boundary condition

$$ u(0,t)=u(1,t),\ u_x(1,t)=\beta u_x(0,t),\ 0\leqslant t\leqslant T,\ \beta\in\mathbb{R}-\{-1,1\} \tag{5.3} $$

and the over deternmination condition

$$ \int_0^1 u(x,t)dx = g(t) \tag{5.4} $$

wehere ${}^{C}\mathcal{D}^{\alpha,\rho}_{0^+}$ the Katugampola fractional derivative of order $0<\alpha<1$, and $r(t),\ \varphi(x),\ f(x,t)$ and $g(t)$ are given function.

5.1 The finite difference scheme

In this section,we payourattention on a numerical approach to generalized Caputo-Katugampola fractional differential equation. The finite difference scheme. from [45] we sub-divide the intervals $[0,1]$ and $[0,T]$ with

$$\begin{aligned} x_i &= (ik)^{\frac{1}{\rho}}\,,\ i=0,1,...,M \\ t_j &= (jh)^{\frac{1}{\rho}}\,,\ j=0,1,...,N \end{aligned}$$

where

$$k=\frac{1}{M},\ h=\frac{T^\rho}{N}$$

are the spatial and temporal step sizes, respectively. We denote u_i^{n+1} the numerical approximation to $u(x_i,t_{n+1}),\ r^{n+1}=r(t_{n+1})$ and $f_i^{n+1}=f(x_i,t_{n+1})$.

1. The initial boundary conditions (5.2)-(5.3), are discretizedas

$$\begin{cases} u_i^0=\varphi_i, \\ u_0^{n+1}=u_M^{n+1}, \\ u_M^{n+1}-u_{M-1}^{n+1}=\beta\left(u_1^{n+1}-u_0^{n+1}\right), \end{cases} \tag{5.5}$$

the over deternmination condition

$$g(t_{n+1})=g^{n+1}=\int_0^1 u(x,t_{n+1})dx=ku_0^{n+1}+k\sum_{i=1}^{M-1}u_i^{n+1},\ n=0,1,...,N-1, \tag{5.6}$$

where

$$u_0^{n+1}=\frac{1}{1+\beta}\left(\beta u_1^{n+1}+u_{M-1}^{n+1}\right).$$

2. The approximation of the Caputo-Katugampola fractional derivative ${}^C\mathcal{D}_t^{\alpha,\rho}$ in (5.1) is given by the following scheme ([45]):

$${}^C\mathcal{D}_t^{\alpha,\rho}u(x_i,t_{n+1})=\frac{h^{1-\alpha}\rho^{\alpha-1}}{\Gamma\left(2-\alpha\right)}\sum_{j=0}^{n}b_j\left(u_i^{j+1}-u_i^j\right), \tag{5.7}$$

where

$$b_j = \frac{t_{j+1}^{1-\rho}}{t_{j+1} - t_j}\left((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}\right), \quad j = 0, 1, ..., n. \tag{5.8}$$

and

$$^{C}\mathcal{D}_t^{\alpha,\rho} u(x_i, t_{n+1}) :=^{C} \mathcal{D}_t^{\alpha,\rho} u_i^{n+1} + C_{\alpha,\rho} h^{2-\alpha}.$$

3. For $u_{xx}(x,t)$, the well known formula is:

$$u_{xx}(x_i, t_{n+1}) = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{k^2} + \gamma k^2. \tag{5.9}$$

Now, By using the time-fractional approximation (5.7) and (5.9) we obtain the following numerical approximation to equation (5.1),

$$\frac{h^{1-\alpha}\rho^{\alpha-1}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j \left(u_i^{j+1} - u_i^j\right) = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{k^2} + r^{n+1} f_i^{n+1},$$

$$-\lambda u_{i+1}^{n+1} + (b_n + 2\lambda) u_i^{n+1} - \lambda u_{i-1}^{n+1} = b_n u_i^n - \sum_{j=0}^{n-1} b_j \left(u_i^{j+1} - u_i^j\right) + \lambda k^2 r^{n+1} f_i^{n+1}, \tag{5.10}$$

where

$$\lambda = \frac{\Gamma(2-\alpha) h^{\alpha-1}}{\rho^{\alpha-1} k^2}.$$

for each $n \in \{0, 1, ..., N-1\}$, and $i \in \{0, 1, ..., M\}$.

- For $n = 0$, and $i = 1, ..., M-1$,

$$-\lambda u_{i+1}^1 + (b_0 + 2\lambda) u_i^1 - \lambda u_{i-1}^1 = b_0 \varphi_i + \lambda k^2 r^1 f_i^1, \tag{5.11}$$

- For $n > 0$, and $i = 1, ..., M-1$,

$$-\lambda u_{i+1}^{n+1} + (b_n + 2\lambda) u_i^{n+1} - \lambda u_{i-1}^{n+1} = \sum_{j=1}^{n} (b_j - b_{j-1}) u_i^j + b_0 \varphi_i + \lambda k^2 r^{n+1} f_i^{n+1}, \tag{5.12}$$

The above equation (5.11) and (5.12) can be written as

$$\begin{cases} \mathbf{A}_0 \mathbf{U}^1 = b_0 \varphi + \mathbf{F}^1 \\ \mathbf{A}_n \mathbf{U}^{n+1} = \sum\limits_{j=1}^{n} (b_j - b_{j-1}) \mathbf{U}^j + b_0 \varphi + \mathbf{F}^{n+1} \end{cases}, \tag{5.13}$$

where,

$$\mathbf{A}_n = \begin{pmatrix} b_n + 2\lambda - \frac{\lambda\beta}{1+\beta} & -\lambda & 0 & \cdots & 0 & -\frac{\lambda}{1+\beta} \\ -\lambda & b_n + 2\lambda & -\lambda & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda & b_n + 2\lambda & -\lambda \\ -\frac{\lambda\beta}{1+\beta} & 0 & \cdots & 0 & -\lambda & b_n + 2\lambda - \frac{\lambda}{1+\beta} \end{pmatrix} \tag{5.14}$$

and

$$\mathbf{U}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{M-1}^n \end{pmatrix}, \ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{M-1} \end{pmatrix}, \ \mathbf{F}^n = \begin{pmatrix} \lambda k^2 r^{n+1} f_1^n \\ \lambda k^2 r^{n+1} f_2^n \\ \vdots \\ \lambda k^2 r^{n+1} f_{M-1}^n \end{pmatrix}.$$

Proof.

Lemma 5.1. *The matrice (5.14) is inversible and the matricial system (5.13) has a unique solution if*

$$T \leq \left(2\frac{C_n}{c_2}\rho^{\alpha-1}\frac{|\beta+1|}{|\beta-1|} \right)^{\frac{1}{\alpha\rho}}. \tag{5.15}$$

where $C_n = \frac{\left(\frac{n+1}{N}\right)^{\frac{1-\rho}{\rho}}}{\left(\frac{n+1}{N}\right)^{\frac{1}{\rho}} - \left(\frac{n}{N}\right)^{\frac{1}{\rho}}}$, $c_2 = \frac{\Gamma(2-\alpha)}{N^{\alpha-1}k^2}$.

□

Proof. Clarely the matrice $\mathbf{A}_n$ is inversible becease it is positive defined i.e,

$$\begin{aligned} \left(\mathbf{U}^n\right)^t \mathbf{A}_n \mathbf{U}^n &= b_n \sum_{i=2}^{M-2} (u_i^n)^2 + \lambda \sum_{i=2}^{M-2} \left(u_{i+1}^n - u_i^n\right)^2 + \frac{\lambda}{2}\left(u_1^n - u_{M-1}^n\right)^2 \\ &+ \left(b_n - \frac{\lambda\beta}{1+\beta} + \frac{\lambda}{2}\right)(u_1^n)^2 + \left(b_n - \frac{\lambda}{1+\beta} + \frac{\lambda}{2}\right)\left(u_{M-1}^n\right)^2, \end{aligned}$$

where $b_n = C_n T^{-\rho} > 0, \ \lambda = \frac{c_2}{\rho^{\alpha-1}} T^{(\alpha-1)\rho} > 0$.
If $b_n - \frac{\lambda\beta}{1+\beta} + \frac{\lambda}{2} \geq 0$ and $b_n - \frac{\beta}{1+\beta} + \frac{\lambda}{2} \geq 0$, then $b_n \geq \frac{\lambda}{2}\frac{|\beta+1|}{|\beta-1|}$. Therefor, we find that

$$T \leq \left(2\frac{C_n}{c_2}\rho^{\alpha-1}\frac{|\beta+1|}{|\beta-1|} \right)^{\frac{1}{\alpha\rho}}.$$

Hence,

$$(\mathbf{U}^n)^t \mathbf{A}_n \mathbf{U}^n > 0.$$

So the system (5.13) admits a unique solution. □

New, integrating equation (5.1) with respect to x from 0 to 1 and using (5.3) and (5.4), we obtain

$$r(t) = \frac{{}^C\mathcal{D}_t^{\alpha,\rho} g(t) + (1-\beta) u_x(0,t)}{\int_0^1 f(x,t)dx} \tag{5.16}$$

see [] .

The finite difference approximation of (5.16) is

$$r^{n+1} = \frac{\frac{h^{1-\alpha}\rho^{\alpha-1}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j \left(g^{j+1} - g^j\right) + \frac{(1-\beta)}{k}\left(u_1^{n+1} - u_0^{n+1}\right)}{(Fin)^{n+1}} \tag{5.17}$$

where $(Fin)^{n+1} = \int_0^1 f(x, t_{n+1})dx = k f_0^{n+1} + k \sum_{i=1}^{M-1} f_i^{n+1},\ n = 0, 1, ..., N-1.$

For $n = 0$,

$$r^1 = \frac{\frac{(1-\beta)}{k}\left(u_1^1 - u_0^1\right)}{(Fin)^1}. \tag{5.18}$$

5.2 Stability analysis of finite difference scheme

Now, we analyze the stability via mathematical induction method, we suppose that $\tilde{u}_i^n$, for $i = 0, 1, 2, ..., M$, and $n = 0, 1, 2, ..., N$ is the approximate solution of (5.11), (5.12), the error $\varepsilon_i^n = \tilde{u}_i^n - u_i^n$. From (5.11) and (5.12), we have

For $n = 0$,

$$-\lambda\varepsilon_{i+1}^1 + (b_0 + 2\lambda)\varepsilon_i^1 - \lambda\varepsilon_{i-1}^1 = b_0(\tilde{\varphi}_i - \varphi_i) + \lambda k^2(\tilde{r}^1\tilde{f}_i^1 - r^1 f_i^1),$$

For $n > 0$,

$$\begin{aligned} -\lambda\varepsilon_{i+1}^{n+1} + (b_n + 2\lambda)\varepsilon_i^{n+1} - \lambda\varepsilon_{i-1}^{n+1} &= \sum_{j=1}^{n} (b_j - b_{j-1})\left(\tilde{u}_i^j - u_i^j\right) \\ &\quad + b_0(\tilde{\varphi}_i - \varphi_i) + \lambda k^2\left(\tilde{r}^{n+1}\tilde{f}_i^{n+1} - r^{n+1} f_i^{n+1}\right). \end{aligned}$$

Let $\left\{u_i^{n+1}, r^{n+1}\right\}$ and $\left\{\tilde{u}_i^{n+1}, \tilde{r}^{n+1}\right\}$ be two solution of the scheme (5.11), (5.12) and (5.17) for the inverse problem (5.1)-(5.4), corresponding to the data $\Psi_i^{n+1} = \{\varphi_i, f_i^{n+1}, g^{n+1}\}$,

Let $\mathcal{H}$ be the set of triples $\{\varphi_i, f_i^{n+1}, g^{n+1}\}$ where the functions φ_i, f_i^{n+1} and g^{n+1} satisfy the assumptions of Theorem 4.1,

for $\phi_i^{n+1} \in \mathcal{H}$ and $\phi_i^{n+1} = \{\varphi_i, g^{n+1}\}$, we define the norm

$$\left\|\phi_i^{n+1}\right\|_{\mathcal{H}} = \|\varphi_i\| + \left\|g^{n+1}\right\|,$$

where

$$\|\varphi_i\| = \max_{0\leq i\leq M-1} |\varphi_i|, \ \left\|g^{n+1}\right\| = \max_{0\leq n\leq N-1} \left|g^{n+1}\right|.$$

for $\Psi_i^{n+1} \in \mathcal{H}$ and $\Psi_i^{n+1} = \{\phi_i^{n+1}, f_i^{n+1}\}$, we define the norm

$$\left\|\Psi_i^{n+1}\right\|_{\mathcal{H}} = \max\left\{\left\|\phi_i^{n+1}\right\|_{\mathcal{H}}, \left\|f_i^{n+1}\right\|_{\mathcal{H}}\right\}$$

where

$$\left\|f_i^{n+1}\right\|_{\mathcal{H}} = \max_{\substack{0\leq i\leq M-1\\ 0\leq n\leq N-1}} \left|f_i^{n+1}\right|.$$

Theorem 5.1. *Let the following assumptions be satisfied*

(A1) $\varphi_0 = \varphi_M,\ \varphi_M - \varphi_{M-1} = \beta(\varphi_1 - \varphi_0)$

(A2) $f_0^{n+1} = f_M^{n+1},\ f_M^{n+1} - f_{M-1}^{n+1} = \beta\left(f_1^{n+1} - f_0^{n+1}\right)$

and there existe a constant M_3 *such that*

$$0 < \left|\int_0^1 f(x, t_{n+1})dx\right|^{-1} = \left|Fin^{n+1}\right|^{-1} < M_3.$$

(A3) g^0 *satisfies the consistency condition* $g^0 = \int_0^1 \varphi(x)dx = \frac{k}{1+\beta}(\beta\varphi_1 + \varphi_{M-1}) + k\sum_{i=1}^{M-1}\varphi_i.$

(A4) $\left\|r^{n+1}\right\| = M_0,\ \left\|f_i^{n+1}\right\| = M_1,\ \left\|g^{n+1}\right\| = M_2,$

If

$$T < \left(\frac{C_n\,|1+\beta|\,\rho^{\alpha-1}}{2c_2kM_1M_3\,|\beta-1|\,(|\beta|+1)}\right)^{\frac{1}{\alpha\rho}}.$$

The discretised scheme (5.11), (5.12) and (5.17) for the inverse problem (5.1)-(5.4) is conditionally stable.

Proof. We have prove that

$$\left\|\varepsilon_i^{n+1}\right\| \leq \mathfrak{C}_n \left\|\Psi_i^{n+1} - \tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}},$$

where $\left\|\varepsilon_i^{n+1}\right\| = \max_{1\leq i\leq M-1}\left|\varepsilon_i^{n+1}\right| = \left|\varepsilon_l^{n+1}\right|$.

- For $n=0$, we have

$$\begin{aligned} b_0\left|\varepsilon_l^1\right| &= -\lambda\left|\varepsilon_l^1\right| + (b_0+2\lambda)\left|\varepsilon_l^1\right| - \lambda\left|\varepsilon_l^1\right| \\ &\leq -\lambda\left|\varepsilon_{l+1}^1\right| + (b_0+2\lambda)\left|\varepsilon_l^1\right| - \lambda\left|\varepsilon_{l-1}^1\right| \\ &\leq \left|-\lambda\varepsilon_{l+1}^1 + (b_0+2\lambda)\varepsilon_l^1 - \lambda\varepsilon_{l-1}^1\right| \\ &\leq b_0\left|\tilde{\varphi}_l - \varphi_l\right| + \lambda k^2\left(\left|\tilde{r}^1\tilde{f}_l^1 - r^1 f_l^1\right|\right) \\ &\leq b_0\left|\tilde{\varphi}_l - \varphi_l\right| + \lambda k^2\left(\left|\tilde{f}_l^1\right|\left|r^1-\tilde{r}^1\right| + \left|r^1\right|\left|\tilde{f}_l^1 - f_l^1\right|\right), \end{aligned}$$

since

$$\begin{aligned} \left|r^1-\tilde{r}^1\right| &= \frac{1}{|1+\beta|}\left(|\beta|\left|f_1^{n+1}-\tilde{f}_1^{n+1}\right| + \left|f_{M-1}^{n+1}-\tilde{f}_{M-1}^{n+1}\right|\right) \\ &\quad + k\frac{\left|r^1\right|}{\left|(Fin)^1\right|}\left|\sum_{i=1}^{M-1}\left(f_i^{n+1}-\tilde{f}_i^{n+1}\right)\right| \\ &\quad + \left|\frac{1-\beta}{k\,(Fin)^1}\right|\left(\left|\varepsilon_1^1\right| + \left|\varepsilon_0^1\right|\right), \end{aligned}$$

and

$$\begin{aligned} \left\|r^1-\tilde{r}^1\right\| &\leq kM_0M_3\left(\frac{1-k}{k} + \frac{|\beta|+1}{|\beta+1|}\right)\left\|f_i^{n+1}-\tilde{f}_i^{n+1}\right\| \\ &\quad + 2M_3\frac{|1-\beta|}{k\,|1+\beta|}\left(|\beta|+1\right)\left|\varepsilon_l^1\right|. \end{aligned}$$

then

$$\begin{aligned} b_0\left|\varepsilon_l^1\right| &\leq b_0\left\|\tilde{\varphi}_i - \varphi_i\right\| \\ &\quad + \lambda k^2 M_0\left[kM_1M_3\left(\frac{1-k}{k} + \frac{|\beta|+1}{|\beta+1|}\right) + 1\right]\left\|f_i^1 - \tilde{f}_i^1\right\| \\ &\quad + 2\lambda k M_1 M_3\frac{|1-\beta|}{|1+\beta|}\left(|\beta|+1\right)\left|\varepsilon_l^1\right|. \end{aligned}$$

If $b_0 - 2\lambda k M_1 M_3\frac{|1-\beta|}{|1+\beta|}\left(|\beta|+1\right) > 0$, i.e $\left(T < \left(\frac{C_0|1+\beta|\rho^{\alpha-1}}{2c_2kM_1M_3|\beta-1|(|\beta|+1)}\right)^{\frac{1}{\alpha\rho}}\right)$ we find that

$$\left|\varepsilon_l^1\right| \leq \mathfrak{C}_0\left\|\Psi_i^{n+1} - \tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}},$$

where $\mathfrak{C}_0 = \frac{b_0+\lambda k^2 M_0\left[kM_1M_3\left(\frac{1-k}{k}+\frac{|\beta|+1}{|\beta+1|}\right)+1\right]}{b_0-2\lambda k M_1 M_3\frac{|1-\beta|}{|1+\beta|}(|\beta|+1)}$.

and

$$\left\|r^1-\tilde{r}^1\right\| \leq \left[\begin{array}{c} kM_0M_3\left(\frac{1-k}{k}+\frac{|\beta|+1}{|\beta+1|}\right) \\ +2M_3\frac{|1-\beta|}{k|1+\beta|}\left(|\beta|+1\right)\mathfrak{C}_0 \end{array}\right]\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}}.$$

- For $n \geq 1$,we assum that

$$\left\|\varepsilon_i^j\right\| \leq \mathfrak{C}_j\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}},\ j=0,1,...,n.$$

$$\begin{aligned} b_n\left|\varepsilon_l^{n+1}\right| &= -\lambda\left|\varepsilon_l^{n+1}\right| + \left(b_n+2\lambda\right)\left|\varepsilon_l^{n+1}\right| - \lambda\left|\varepsilon_l^{n+1}\right| \\ &\leq -\lambda\left|\varepsilon_{l+1}^{n+1}\right| + \left(b_n+2\lambda\right)\left|\varepsilon_l^{n+1}\right| - \lambda\left|\varepsilon_{l-1}^{n+1}\right| \\ &\leq \left|-\lambda\varepsilon_{l+1}^{n+1} + \left(b_n+2\lambda\right)\varepsilon_l^{n+1} - \lambda\varepsilon_{l-1}^{n+1}\right| \\ &\leq \left|\sum_{j=1}^{n}\left(b_j-b_{j-1}\right)\varepsilon_l^j + b_0\left(\varphi_l-\tilde{\varphi}_l\right) + \lambda k^2\left(r^{n+1}f_l^{n+1}-\tilde{r}^{n+1}\tilde{f}_l^{n+1}\right)\right| \\ &\leq \left|\sum_{j=1}^{n}\left(b_j-b_{j-1}\right)\right|\left|\varepsilon_l^j\right| + b_0\left|\varphi_l-\tilde{\varphi}_l\right| + \lambda k^2\left|r^{n+1}f_l^{n+1}-\tilde{r}^{n+1}\tilde{f}_l^{n+1}\right|, \end{aligned}$$

which gives

$$\begin{aligned} b_n\left|\varepsilon_l^{n+1}\right| \leq\ & \left|b_n-b_0\right|\left|\varepsilon_l^j\right| + b_0\left|\varphi_l-\tilde{\varphi}_l\right| \\ & +\lambda k^2\left(\left|\tilde{f}_l^{n+1}\right|\left|\tilde{r}^{n+1}-r^{n+1}\right| + \left|r^{n+1}\right|\left|\tilde{f}_l^{n+1}-f_l^{n+1}\right|\right). \end{aligned} \tag{5.19}$$

since

$$
\begin{aligned}
& \left|r^{n+1}-\tilde{r}^{n+1}\right| \\
= \; & k\frac{\left|r^{n+1}\right|}{\left|(Fin)^{n+1}\right|}\frac{1}{|1+\beta|}\left(|\beta|\left|f_1^{n+1}-\tilde{f}_1^{n+1}\right|+\left|f_{M-1}^{n+1}-\tilde{f}_{M-1}^{n+1}\right|\right) \\
& +k\frac{\left|r^{n+1}\right|}{\left|(Fin)^{n+1}\right|}\left|\sum_{i=1}^{M-1}\left(f_i^{n+1}-\tilde{f}_i^{n+1}\right)\right| \\
& +\frac{h^{1-\alpha}\rho^{\alpha-1}}{\left|(Fin)^{n+1}\right|\Gamma(2-\alpha)}\left[\begin{array}{l} b_{n-1}\left|g^{n+1}-\tilde{g}^{n+1}\right| \\ +b_0\frac{k}{|1+\beta|}\left(|\beta|\left|\varphi_1-\tilde{\varphi}_1\right|+\left|\varphi_{M-1}-\tilde{\varphi}_{M-1}\right|+\sum\limits_{i=1}^{M-1}\left|\varphi_i-\tilde{\varphi}_i\right|\right) \\ +\sum\limits_{j=1}^{n}\left|b_{j-1}-b_j\right|\left|g^j-\tilde{g}^j\right| \end{array}\right] \\
& +2M_3\frac{|1-\beta|}{k|1+\beta|}\left(|\beta|+1\right)\left|\varepsilon_l^{n+1}\right|.
\end{aligned}
$$

then

$$
\begin{aligned}
& \left\|r^{n+1}-\tilde{r}^{n+1}\right\| \\
\leq \; & k\frac{\left|r^{n+1}\right|}{\left|(Fin)^{n+1}\right|}\left(\frac{2|\beta|}{|1+\beta|}+(M-1)\right)\left\|f_i^{n+1}-\tilde{f}_i^{n+1}\right\| \\
& +\frac{h^{1-\alpha}\rho^{\alpha-1}}{\left|(Fin)^{n+1}\right|\Gamma(2-\alpha)}\left[\begin{array}{l} b_{n-1}\left\|g^{n+1}-\tilde{g}^{n+1}\right\| \\ +b_0\frac{k}{|1+\beta|}\left(|\beta|+M\right)\left\|\varphi_i-\tilde{\varphi}_i\right\| \\ +\left|b_{n-1}-b_0\right|\left\|g^{n+1}-\tilde{g}^{n+1}\right\| \end{array}\right] \\
& +2M_3\frac{|1-\beta|}{k|1+\beta|}\left(|\beta|+1\right)\left|\varepsilon_l^{n+1}\right| \\
\leq \; & kM_0M_3\left(\frac{2|\beta|}{|1+\beta|}+\left(\frac{1-k}{k}\right)\right)\left\|f_i^{n+1}-\tilde{f}_i^{n+1}\right\| \\
& +\frac{h^{1-\alpha}\rho^{\alpha-1}}{\Gamma(2-\alpha)}M_3\left[\begin{array}{l} \left(2b_{n-1}-b_0\right)\left\|g^{n+1}-\tilde{g}^{n+1}\right\| \\ +b_0\frac{k}{|1+\beta|}\left(|\beta|+\frac{1}{k}\right)\left\|\varphi_i-\tilde{\varphi}_i\right\| \end{array}\right] \\
& +2M_3\frac{|1-\beta|}{k|1+\beta|}\left(|\beta|+1\right)\left|\varepsilon_l^{n+1}\right| \\
\leq \; & kM_0M_3\left(\frac{2|\beta|}{|1+\beta|}+\left(\frac{1-k}{k}\right)\right)\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}} \\
& +M_4\left\|\phi_i^{n+1}-\tilde{\phi}_i^{n+1}\right\|_{\mathcal{H}}+2M_3\frac{|1-\beta|}{k|1+\beta|}\left(|\beta|+1\right)\left|\varepsilon_l^{n+1}\right|
\end{aligned}
$$

where $M_4=\max\left\{2b_{n-1}-b_0, b_0\frac{k}{|1+\beta|}\left(|\beta|+\frac{1}{k}\right)\right\}$.

this is amplies that

$$\begin{aligned} \left\|r^{n+1}-\tilde{r}^{n+1}\right\| &\leq \left[kM_0M_3\left(\frac{2|\beta|}{|1+\beta|}+\left(\frac{1-k}{k}\right)\right)+M_4\right]\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}} \\ &\quad +2M_3\frac{|1-\beta|}{k|1+\beta|}(|\beta|+1)\left|\varepsilon_l^{n+1}\right|. \end{aligned} \tag{5.20}$$

According to (5.19), we imply

$$\begin{aligned} b_n\left|\varepsilon_l^{n+1}\right| &\leq |b_n-b_0|\left|\varepsilon_l^{j}\right|+b_0|\varphi_l-\tilde{\varphi}_l| \\ &\quad +\lambda k^2\left|\tilde{f}_l^{n+1}\right|\left(\begin{array}{c} \left[kM_0M_3\left(\frac{2|\beta|}{|1+\beta|}+\left(\frac{1-k}{k}\right)\right)+M_4\right]\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}} \\ +2M_3\frac{|1-\beta|}{k|1+\beta|}(|\beta|+1)\left|\varepsilon_l^{n+1}\right| \end{array}\right) \\ &\quad +\lambda k^2\left|r^{n+1}\right|\left|\tilde{f}_l^{n+1}-f_l^{n+1}\right| \\ &\leq (b_n-b_0)\mathfrak{C}_j\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}}+b_0\left\|\phi_i^{n+1}-\tilde{\phi}_i^{n+1}\right\|_{\mathcal{H}} \\ &\quad +\lambda k^2M_1\left(\begin{array}{c} \left[kM_0M_3\left(\frac{2|\beta|}{|1+\beta|}+\left(\frac{1-k}{k}\right)\right)+M_4\right]\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}} \\ +2M_3\frac{|1-\beta|}{k|1+\beta|}(|\beta|+1)\left|\varepsilon_l^{n+1}\right| \end{array}\right) \\ &\quad +\lambda k^2M_0\left\|\tilde{f}_l^{n+1}-f_l^{n+1}\right\| \\ &\leq M_5\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}} \\ &\quad +2\lambda k^2M_1M_3\frac{|1-\beta|}{k|1+\beta|}(|\beta|+1)\left|\varepsilon_l^{n+1}\right|. \end{aligned}$$

where $M_5=\left[(b_n-b_0)\mathfrak{C}_j+b_0+\lambda k^2M_1\left(kM_0M_3\left(\frac{2|\beta|}{|1+\beta|}+\left(\frac{1-k}{k}\right)\right)+M_4\right)+\lambda k^2M_0\right]$,

this is amplies that

$$\left(b_n-2\lambda k^2M_1M_3\frac{|1-\beta|}{k|1+\beta|}(|\beta|+1)\right)\left|\varepsilon_l^{n+1}\right|\leq M_5\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}},$$

If $b_n-2\lambda k^2M_1M_3\frac{|1-\beta|}{k|1+\beta|}(|\beta|+1)>0,$ i.e $\left(T<\left(\frac{C_n|1+\beta|\rho^{\alpha-1}}{2c_2kM_1M_3|\beta-1|(|\beta|+1)}\right)^{\frac{1}{\alpha\rho}}\right)$ we find that

$$\left|\varepsilon_l^{n+1}\right|\leq\mathfrak{C}_n\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}},$$

where $\mathfrak{C}_n=\frac{M_5}{b_n-2\lambda k^2M_1M_3\frac{|1-\beta|}{k|1+\beta|}(|\beta|+1)}$.

According to (5.20), we imply

$$\begin{aligned}
\left\|r^{n+1}-\tilde{r}^{n+1}\right\| &\leq \left[kM_0M_3\left(\frac{2\left|\beta\right|}{\left|1+\beta\right|}+\left(\frac{1-k}{k}\right)\right)+M_4\right]\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}} \\
&\quad +2M_3\frac{\left|1-\beta\right|}{k\left|1+\beta\right|}\left(\left|\beta\right|+1\right)\mathfrak{C}_n\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}} \\
&\leq M_6\left\|\Psi_i^{n+1}-\tilde{\Psi}_i^{n+1}\right\|_{\mathcal{H}}.
\end{aligned}$$

where $M_6=kM_0M_3\left(\frac{2|\beta|}{|1+\beta|}+\left(\frac{1-k}{k}\right)\right)+M_4+2M_3\frac{|1-\beta|}{k|1+\beta|}\left(\left|\beta\right|+1\right)\mathfrak{C}_n$.

The proof is complet. □

5.3 Convergence of the approximate scheme

In this section, we discuss the convergence of the approximate scheme (5.11), and (5.12).

Let $u\left(x_i,t_n\right)$ be the exact solution of the time fractional diffusion equation (5.1)-(5.2)-(5.3) at mesh points $\left(x_i,t_n\right)$ where $\left(i=0,1,2,\ldots,M;n=0,1,2,\ldots,N\right)$.

Define $e_i^n=u\left(x_i,t_n\right)-u_i^n$, and $e^n=\left(e_1^n,e_2^n,\ldots,e_{M-1}^n\right)^t$. Using $e^0=\left(0,0,\ldots,0\right)^t$. Substitution $u_i^n=u\left(x_i,t_n\right)-e_i^n$ into (5.11), (5.12) leads to:

1. For $n=0$, the approximate scheme (5.11), gives

$$\begin{aligned}
-\lambda e_{i+1}^1+\left(b_0+2\lambda\right)e_i^1-\lambda e_{i-1}^1 &= -\lambda u\left(x_{i+1},t_1\right)+\left(2\lambda+b_0\right)u\left(x_i,t_1\right) \\
&- \lambda u\left(x_{i-1},t_1\right)-b_0\varphi_i-\lambda k^2r^1f_i^1 \\
&= R_i^1.
\end{aligned}$$

2. For $n>0$, the approximate scheme (5.12) gives

$$\begin{aligned}
&-\lambda e_{i+1}^{n+1}+\left(2\lambda+b_n\right)e_i^{n+1}-\lambda e_{i-1}^{n+1} \\
&= \sum_{j=1}^{n}\left(b_j-b_{j-1}\right)e_i^j+R_i^{n+1},
\end{aligned}$$

where

$$\begin{aligned} R_i^{n+1} &= -\sum_{j=1}^{n}(b_j-b_{j-1})\,u(x_i,t_j) \\ &\quad -\lambda u(x_{i+1},t_{n+1})+(2\lambda+b_n)\,u(x_i,t_{n+1})-\lambda u(x_{i-1},t_{n+1}) \\ &\quad -b_0\varphi_i-\lambda k^2 r^{n+1} f_i^{n+1} \end{aligned}$$

then

$$\begin{aligned} R_i^{n+1} &= \sum_{j=0}^{n} b_j\left(u(x_i,t_{j+1})-u(x_i,t_j)\right) \\ &\quad -\lambda\left(u(x_{i+1},t_{n+1})-2u(x_i,t_{n+1})+u(x_{i-1},t_{n+1})\right) \\ &\quad -\lambda k^2 r^{n+1} f_i^{n+1}. \end{aligned}$$

From (5.1) we have

$$\begin{aligned} R_i^{n+1} &= \lambda k^2\left({}^C\mathcal{D}_t^{\alpha,\rho}u(x_i,t_{n+1})-u_{xx}(x_i,t_{n+1})-r^{n+1}f_i^{n+1}-C_{\alpha,\rho}h^{2-\alpha}+\gamma k^2\right) \\ &= \lambda k^2\left(-C_{\alpha,\rho}h^{2-\alpha}+\gamma k^2\right) \\ &= \frac{\Gamma(2-\alpha)}{\rho^{\alpha-1}}h^{\alpha-1}\left(-C_{\alpha,\rho}h^{2-\alpha}+\gamma k^2\right) \end{aligned}$$

Hence, there exist $C^1_{\alpha,\rho}>0$, such that

$$\left|R_i^{n+1}\right|\le C^1_{\alpha,\rho}h^{\alpha-1}\left(h^{2-\alpha}+k^2\right),$$

where $C^1_{\alpha,\rho}=\frac{\Gamma(2-\alpha)}{\rho^{\alpha-1}}\max\left\{\left|C_{\alpha,\rho}\right|,\left|\gamma\right|\right\}.$

Lemma 5.2. *For $n=0,1,2,\ldots,N$, we have*

$$\left\|e_i^{n+1}\right\|\le C^2_{\alpha,\rho}b_n^{-1}h^{\alpha-1}\left(h^{2-\alpha}+k^2\right).$$

Proof. Let $\left\|e_i^{n+1}\right\|=\left|e_l^{n+1}\right|=\max_{1\le i\le M-1}\left|e_i^{n+1}\right|$, and $\left|e_l^1\right|=\max_{1\le i\le M-1}\left|e_i^1\right|.$

For $n=0$, we get

$$\begin{aligned} b_0\left|e_l^1\right| &= -\lambda\left|e_l^1\right|+(b_0+2\lambda)\left|e_l^1\right|-\lambda\left|e_l^1\right| \\ &\leq -\lambda\left|e_{l+1}^1\right|+(b_0+2\lambda)\left|e_l^1\right|-\lambda\left|e_{l-1}^1\right| \end{aligned}$$

imply

$$\begin{aligned} \left|e_l^1\right| &\leqslant b_0^{-1}\left|R_i^1\right| \\ \left|e_l^1\right| &\leqslant b_0^{-1}C_{\alpha,\rho}^1 h^{\alpha-1}\left(h^{2-\alpha}+k^2\right). \end{aligned}$$

For $n>0$, suppose that $\left|e_l^j\right| \leq C_{\alpha,\rho}^1 b_{j-1}^{-1} h^{\alpha-1}\left(h^{2-\alpha}+k^2\right),\ (j=1,\ldots,n)$, we get

$$\left|e_l^{n+1}\right| \leqslant C_{\alpha,\rho}^1 b_n^{-1} h^{\alpha-1}\left(b_n b_0^{-1}\right)\left(h^{2-\alpha}+k^2\right).$$

because $b_{j-1}^{-1}\leq b_0^{-1}$ for $j=1,\ldots,n$, then

$$\left|e_l^{n+1}\right| \leqslant C_{\alpha,\rho}^2 b_n^{-1} h^{\alpha-1}\left(h^{2-\alpha}+k^2\right),$$

where

$$C_{\alpha,\rho}^2 = \begin{cases} C_{\alpha,\rho}^1, & \text{if } n=0 \\ C_{\alpha,\rho}^1\left(b_n b_0^{-1}\right), & \text{if } n\geq 1 \end{cases}.$$

□

We can prove that $\lim\limits_{n\to\infty}\frac{b_n^{-1}}{\left(\frac{t_0^\rho}{h}+n\right)^{\alpha-1}}=0$, there exist a constant $\zeta>0$ suth that

$$\frac{b_n^{-1}}{\left(\frac{t_0^\rho}{h}+n\right)^{\alpha-1}} \leqslant \zeta,$$

then

$$\left|e_l^{n+1}\right| \leqslant C_{\alpha,\rho}^2\zeta\left(\frac{t_0^\rho}{h}+n\right)^{\alpha-1} h^{\alpha-1}\left(h^{2-\alpha}+k^2\right)$$

because

$$\left(\frac{t_0^\rho}{h}+n\right)^{\alpha-1} h^{\alpha-1} = t_n^{\rho(\alpha-1)} \leq T^{\rho(\alpha-1)}.$$

Is finite we have

$$\left|u_i^n - u\left(x_i, t_n\right)\right| \leq C_{\alpha,\rho}^3\left(h^{2-\alpha} + k^2\right)$$

Then we obtain the following theorem:

Theorem 5.2. *Let u_i^n be the approximate value of $u(x_i, t_n)$ computed by use of the difference scheme (5.11) and (5.12). Then there is a positive constant $C_{\alpha,\rho}^3$, such that*

$$\left|u_i^n - u\left(x_i, t_n\right)\right| \leq C_{\alpha,\rho}^3\left(h^{2-\alpha} + k^2\right).$$

5.4 Illustrative examples

In this section, we present some examples to illustrate the usefulness of our main results.

Consider the inverse problem (5.1)-(5.4)

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\alpha,\rho}(u(x,t)) = u_{xx}(x,t) + r(t)f(x,t), \ \ 0 < x < 1, \\ u(0,t) = u(1,t), \ u_x(1,t) = \beta u_x(0,t), \quad 0 \leqslant t \leqslant T, \ \beta \in \mathbb{R} - \{-1, 1\} \\ u(x,0) = \varphi(x), \qquad\qquad\qquad\qquad 0 < x < 1, \\ \int_0^1 u(x,t)dx = g(t), \end{cases}$$

$$r\left(t\right) = \frac{{}^C\mathcal{D}_t^{\alpha,\rho} g\left(t\right) + \left(1-\beta\right) u_x\left(0,t\right)}{\int_0^1 f(x,t)dx}.$$

with

$$\begin{cases} f\left(x,t\right) = \left(1 - \left(1-\beta\right)x\right)\sin\left(2\pi x\right)\left(\frac{\rho^\alpha}{\Gamma(2-\alpha)}.\frac{t^{\rho(1-\alpha)}}{t^\alpha+2} + \left(2\pi\right)^2 \frac{t^\rho+1}{t^\alpha+2}\right) \\ \qquad -4\pi\left(\beta-1\right)\frac{t^\rho+1}{t^\alpha+2}\cos\left(2\pi x\right) \\ \varphi\left(x\right) = \left(1-\left(1-\beta\right)x\right)\sin\left(2\pi x\right) \\ g\left(x\right) = \frac{1-\beta}{2\pi}\left(t^\rho+1\right) \end{cases}$$

It is easy to Check that the exacte solution is

$$\left\{r\left(t\right), u\left(x,t\right)\right\} = \left\{t^\alpha + 2, \left(1-\left(1-\beta\right)x\right)\sin\left(2\pi x\right)\left(t^\rho+1\right)\right\}.$$

The system of equation (**??**), (**??**) and (**??**), (**??**) can be solved the Gauss elimination method and $u_i^{n+1(s+1)}$ is determined. If the differnce of values between two iterations reaches the perscribted tolerance, the iteration is stopped and we accept the corresponding values $r^{n+1(s+1)}$, $u_i^{n+1(s+1)}$, $i = 1, 2, \ldots, M$. as r^{j+1}, u_i^{j+1}, $i = 1, 2, \ldots, M$. on the $(j+1)\,th$ time step, respectively. In virtue of this itiration, we can move from level j to level $j+1$.

Figures 5.1,5.2,...5.11 and 5.12 represent the comparison between the analytical solution and its approximation for different values of h.

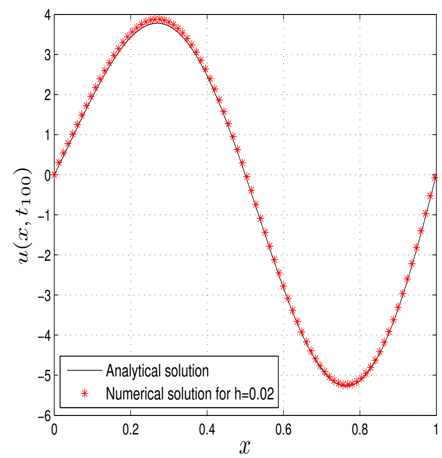

Figure 5.1: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.9$, $rho = 2$, $beta = 2$, $k = 0.002$ and $h = 0.02$.

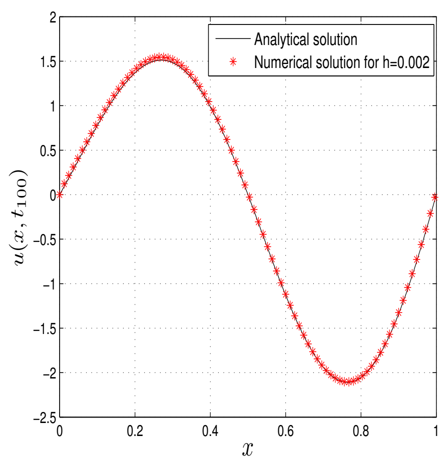

Figure 5.2: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.9$, $rho = 2$, $beta = 2$, $k = 0.002$ and $h = 0.002$.

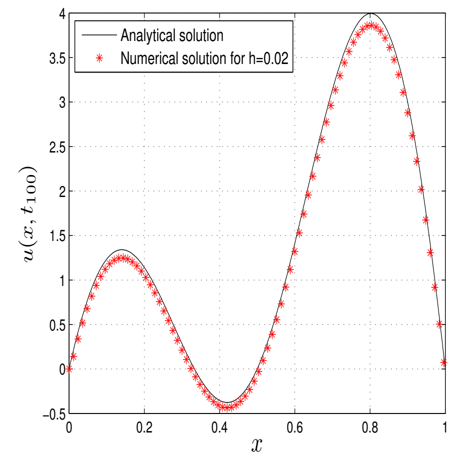

Figure 5.3: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.9$, $rho = 2$, $beta = -2$, $k = 0.002$ and $h = 0.02$.

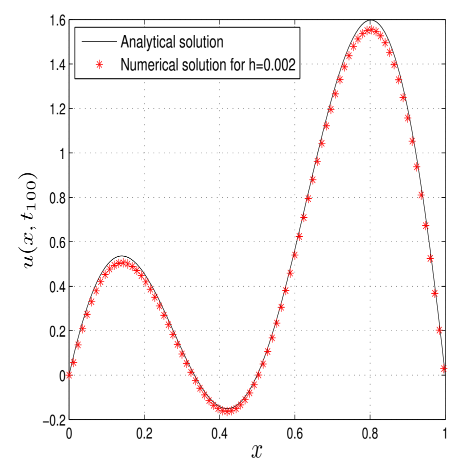

Figure 5.4: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.9$, $rho = 2$, $beta = -2$, $k = 0.002$ and $h = 0.002$.

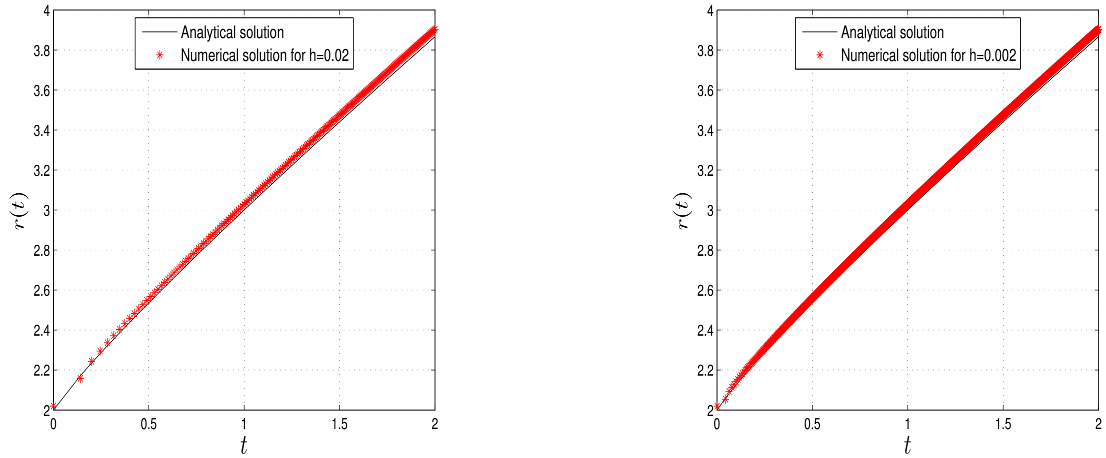

Figure 5.5: Graphical comparison between the analytical solution of $r(t)$ and it approximation with $\alpha = 0.9$, $rho = 2$, $beta = 2$, $k = 0.002$ and $h = 0.02$.

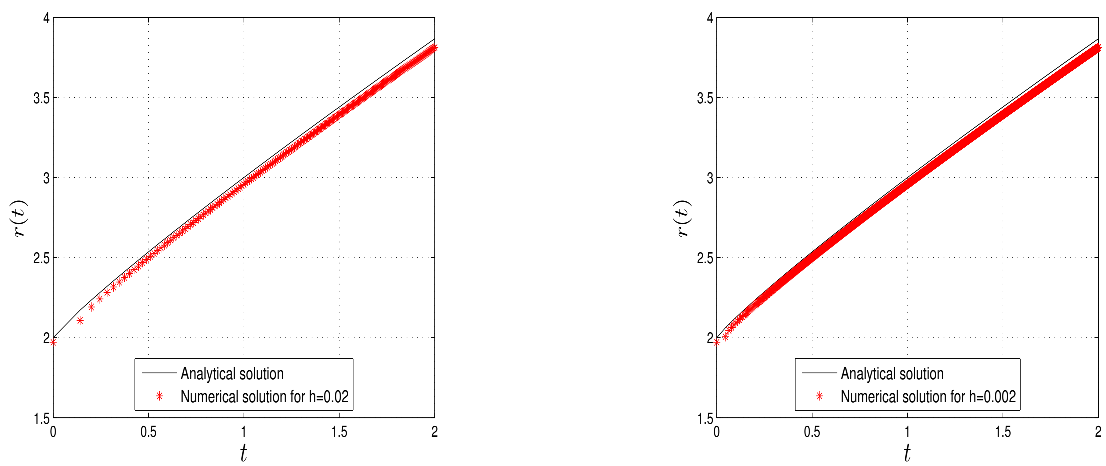

Figure 5.6: Graphical comparison between the analytical solution of $r(t)$ and it approximation with $\alpha = 0.9$, $rho = 2$, $beta = -2$, $k = 0.002$ and $h = 0.02$.

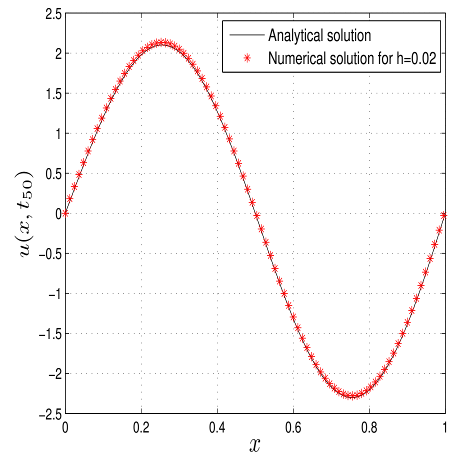

Figure 5.7: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.5$, $rho = 3$, $beta = 1.2$, $k = 0.002$ and $h = 0.02$.

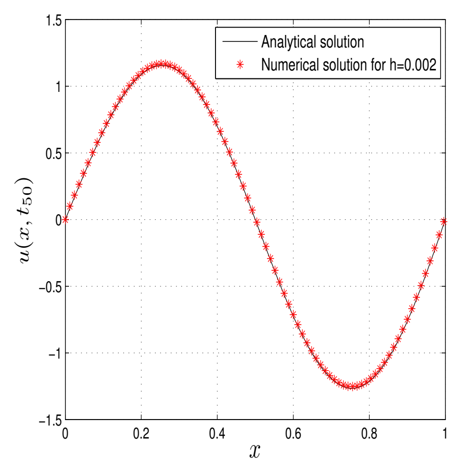

Figure 5.8: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.5$, $rho = 3$, $beta = 1.2$, $k = 0.002$ and $h = 0.02$.

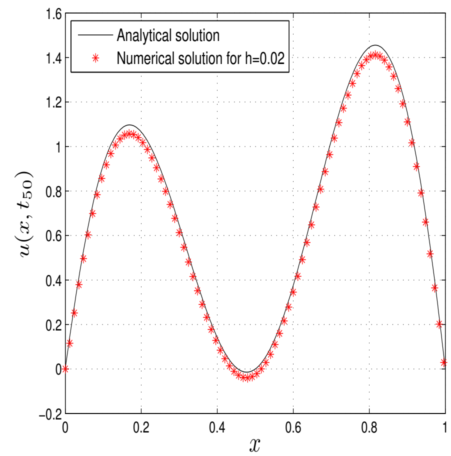

Figure 5.9: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.5$, $rho = 3$, $beta = -1.2$, $k = 0.002$ and $h = 0.02$.

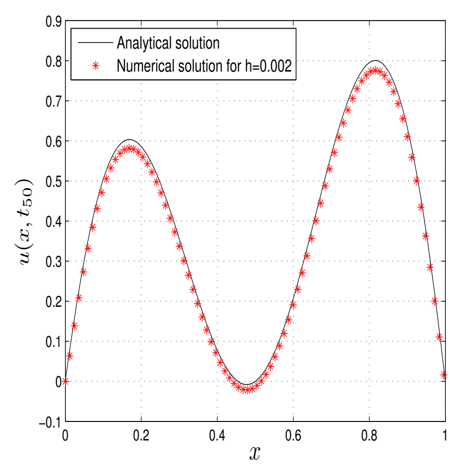

Figure 5.10: Graphical comparison between the analytical solution and it approximation with $\alpha = 0.5$, $rho = 3$, $beta = -1.2$, $k = 0.002$ and $h = 0.002$.

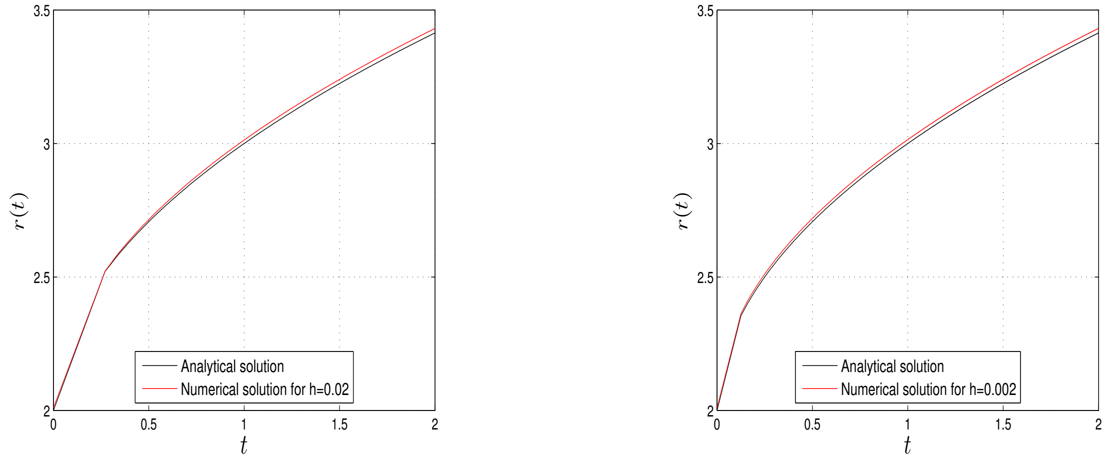

Figure 5.11: Graphical comparison between the analytical solution of $r(t)$ and it approximation with $\alpha = 0.5$, $rho = 3$, $beta = 1.2$, $k = 0.002$ and $h = 0.02$.

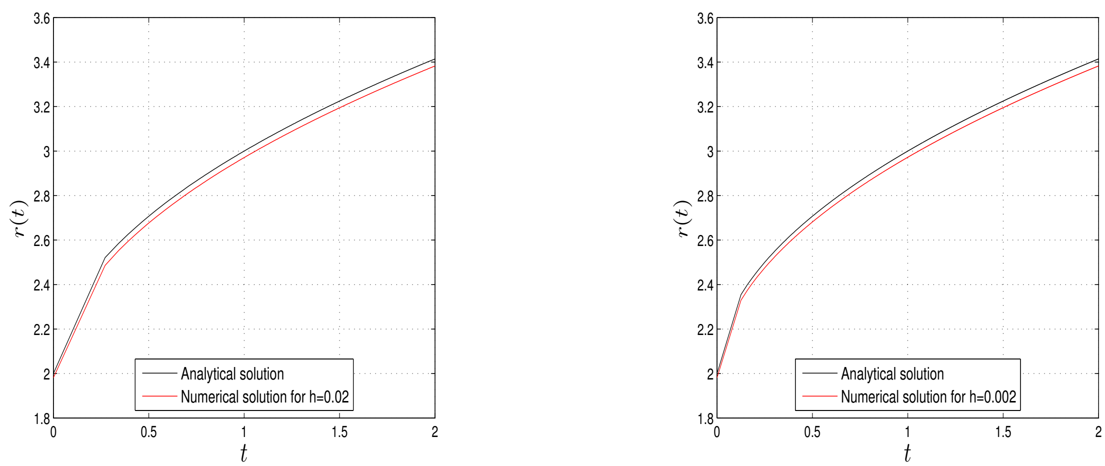

Figure 5.12: Graphical comparison between the analytical solution of $r(t)$ and it approximation with $\alpha = 0.5$, $rho = 3$, $beta = -1.2$, $k = 0.002$ and $h = 0.02$.

CHAPTER 6

A NUMERICAL METHOD FOR SOLVING AN INVERSE TIME-DEPENDENT COEFFICIENT PROBLEM

In this chapter, we propose a numerical method for solving an inverse time-dependent coefficient problem associated with a time-fractional reaction-diffusion equation subject to nonlocal boundary and overdetermination conditions. The time-fractional derivative is considered in the conformable sense. By employing the Shifted Legendre collocation method, the inverse problem is transformed into a linear system of first-order differential equations, which is then solved using the Backward Euler method. Through two illustrative examples, we conduct a comparative analysis between the proposed algorithm and existing numerical methods from the literature. The results demonstrate that our approach achieves highly accurate approximations using a relatively small number of collocation points.

6.1 Shifted Legendre polynomials of the first kind

To use the polynomials given in Section 1.5 on the interval $[0,1]$, we define the so-called shifted Legendre polynomials of the first kind by introducing the following change of variable:

$$z = 2x - 1 \text{ ou } x = \frac{1}{2}(z+1).$$

In this case, the shifted Legendre polynomials $P_n^*(x)$ of order n in x are defined on $[0,1]$ by:

$$P_n^*(x) = P_n(z) = P_n(2x-1). \tag{6.1}$$

With (6.1) and Proposition 1.2, the shifted Legendre polynomials of the first kind $P_n^*(x)$ verify the following recurrence formula, see [13]:

$$\begin{cases} P_0^*(x) = 1, \\ P_1^*(x) = 2x - 1, \\ P_{n+1}^*(x) = \frac{2n+1}{n+1}(2x-1)P_n^*(x) - \frac{n}{n+1}P_{n-1}^*(x), \; n \in \mathbb{N}^*. \end{cases} \tag{6.2}$$

Using (6.1) and (1.12), we obtain the explicit form of the shifted Legendre polynomials of the first kind $P_n^*(x)$ of degree n in x given by:

$$P_n^*(x) = \sum_{k=0}^{n} (-1)^{n+k} \frac{\Gamma(n+k+1)}{\Gamma(n-k+1)\left(\Gamma(k+1)\right)^2} x^k, \;\; n \in \mathbb{N}, \tag{6.3}$$

where $\Gamma(\cdot)$ is Euler's Gamma function. We note that

$$\begin{cases} P_n^*(0) = (-1)^n, & (P_n^*(0))' = (-1)^{n-1} n(n+1), \\ P_n^*(1) = 1, & (P_n^*(1))' = n(n+1). \end{cases} \tag{6.4}$$

According to (6.1) and Lemma 1.3, the polynomials $P_n^*(x)$ are orthogonal on the interval $[0,1]$, that is:

$$\left\langle P_i^*(x), P_j^*(x) \right\rangle = \int_0^1 P_i^*(x) P_j^*(x)\, dx = \begin{cases} \frac{1}{2i+1}, & \text{if } \; i = j, \\ 0, & \text{if } \; i \neq j. \end{cases} \tag{6.5}$$

Let $\Phi \in L^2(0,1)$ be expressed in terms of shifted Legendre polynomials of the first kind such that

$$\Phi(x) = \sum_{i=0}^{\infty} c_i P_i^*(x), \tag{6.6}$$

where the coefficients c_i ; $i \in \mathbb{N}$ are given by:

$$c_i = (2i+1) \int_0^1 \Phi(x) P_i^*(x)\, dx. \tag{6.7}$$

In practice, only the first shifted Legendre polynomials $(m+1)$-terms are considered. Then, we have:

$$\Phi_m(x) = \sum_{i=0}^{m} c_i P_i^*(x). \tag{6.8}$$

6.2 Legendre collocation method

In this section, we apply Legendre collocation method to the inverse time-dependent coefficient problem (4.1)- (4.4). Let $w_m(x,t)$ be the approximation of $w(x,t)$ given in the following form:

$$w_m(x,t) = \sum_{i=0}^{m} c_i(t) P_i^*(x), \tag{6.9}$$

By replacing (6.9) into the equation (4.1). Using (4.3)-(4.4) and (1.9), we obtain:

$$\begin{cases} \sum_{i=0}^{m} t^{1-\alpha} c_i'(t) P_i^*(x) = \sum_{i=0}^{m} c_i(t) R_i(x,t) + S(x,t), \\ p(t) = \frac{1}{E(t)} \left[(\beta - 1) \sum_{i=0}^{m} c_i(t) (P_i^*)'(0) + \int_0^1 S(x,t)\,dx - \mathcal{D}_t^{(\alpha)} E(t) \right]. \end{cases} \tag{6.10}$$

We put in (6.10) $x = x_s, s = 1, \ldots, m-1$ the roots of the Legendre polynomial $P_{m-1}(x)$, we have

$$\begin{cases} \sum_{i=0}^{m} t^{1-\alpha} c_i'(t) P_i^*(x_s) = \sum_{i=0}^{m} c_i(t) R_i(x_s,t) + S(x_s,t), \\ p(t) = \frac{1}{E(t)} \left[(\beta - 1) \sum_{i=0}^{m} c_i(t) (P_i^*)'(0) + \int_0^1 S(x,t)\,dx - \mathcal{D}_t^{(\alpha)} E(t) \right]. \end{cases} \tag{6.11}$$

where

$$R_i(x_s,t) = (P_i^*)''(x_s) - p(t) P_i^*(x_s), \; i = 0,1,\ldots,m, \; s = 1,\ldots,m-1. \tag{6.12}$$

From (6.9), (4.3) and (6.4), we obtain

$$\sum_{i=0}^{m} \left[1 - (-1)^i\right] c_i(t) = 0 \tag{6.13}$$

and

$$\sum_{i=1}^{m}\left[1-\beta\left(-1\right)^{i-1}\right]i\left(i+1\right)c_i\left(t\right)=0. \tag{6.14}$$

We introduce the vectors

$$\begin{aligned} X\left(t\right) &= \left(c_0\left(t\right),c_1\left(t\right),\ldots,c_m\left(t\right)\right)^T, \\ \dot{X}\left(t\right) &= \left(c_0'\left(t\right),c_1'\left(t\right),\ldots,c_m'\left(t\right)\right)^T, \\ F\left(t\right) &= \left(S\left(x_1,t\right),S\left(x_1,t\right),\ldots,S\left(x_{m-1},t\right),0,0\right)^T. \end{aligned} \tag{6.15}$$

By combining equations (6.11), (6.12), (6.13) and (6.14), we find the following matrix form

$$\begin{cases} A\left(t\right)\dot{X}\left(t\right)=B\left(t\right)X\left(t\right)+F\left(t\right), \\ X\left(0\right)=\left(c_0\left(0\right),c_1\left(0\right),\ldots,c_m\left(0\right)\right)^T, \end{cases}$$
$$p\left(t\right)=\frac{1}{E\left(t\right)}\left[\left(\beta-1\right)\sum_{i=0}^{m}c_i\left(t\right)\left(P_i^*\right)'\left(0\right)+\int_0^1 S\left(x,t\right)dx-\mathcal{D}_t^{(\alpha)}E\left(t\right)\right]. \tag{6.16}$$

where $A\left(t\right)$ is the damping matrix given by

$$A\left(t\right)=t^{1-\alpha}\begin{bmatrix} P_0\left(x_1\right) & P_1\left(x_1\right) & \ldots & P_m\left(x_1\right) \\ P_0\left(x_2\right) & P_1\left(x_2\right) & \ldots & P_m\left(x_2\right) \\ \vdots & \ddots & \vdots & \vdots \\ P_0\left(x_{m-1}\right) & P_1\left(x_{m-1}\right) & \ldots & P_m\left(x_{m-1}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.17}$$

and $B(t)$ is the stiffness matrix given by

$$B(t) = \begin{bmatrix} R_0(x_1,t) & R_1(x_1,t) & \dots & R_m(x_1,t) \\ R_0(x_2,t) & R_1(x_2,t) & \dots & R_m(x_2,t) \\ \vdots & \ddots & \vdots & \vdots \\ R_0(x_{m-1},t) & R_1(x_{m-1},t) & \dots & R_m(x_{m-1},t) \\ 0 & 2 & \dots & 1-(-1)^m \\ 0 & 0 & \dots & \left(1-\beta(-1)^{m-1}\right) m(m+1) \end{bmatrix} \tag{6.18}$$

6.2.1 Backward Euler method

For positive integer N, $\Delta t = T/N$, denotes the step size of the variable t. So we define $t_j = j\Delta t$ in which $j = 0, 1, \dots, N$, and we introduce the following notations $c_i(t_j) = c_i^j$, $F^j = (S(x_1,t_j), \dots, S(x_{m-1},t_j))^T$ and $X^j = \left(c_0^j, c_1^j, \dots, c_m^j\right)^T$. The system (6.16) is discretized in time by the backward Euler method [35] and takes the following linear system

$$\begin{cases} [A(t_{j+1}) - \Delta t B(t_{j+1})] X^{j+1} = A(t_{j+1}) X^j + \Delta t F^{j+1}, \; j = 0, 1, \dots, N-1, \\ p(t_{j+1}) = \frac{1}{E(t_{j+1})} \left[(\beta - 1) \sum_{i=0}^{m} c_i(t_{j+1}) (P_i^*)'(0) + \int_0^1 S(x, t_{j+1}) \, dx - \mathcal{D}_t^{(\alpha)} E(t_{j+1}) \right], \\ X^0 = \left(c_0^0, c_1^0, \dots, c_m^0\right)^T. \end{cases} \tag{6.19}$$

Algorithm 1 (Algorithm of the Method)

1: **Initializations:**

1. Give the values of T, α and β.
2. Give the values of time step $\tau = T/N_t$ and the step value of the space $h = 1/N_x$.
3. Give the initial condition $\varphi(x)$ and the reaction coefficient $p_{ex}(t)$.
4. Give the source term $S(x,t)$ and the integral $\int_0^1 S(x,t)\,dx$.
5. Give the functions $E(t)$ and $\mathcal{D}^{(\alpha)}E(t)$.
6. Give the shifted Legendre polynomials $P_0^*(x), P_1^*(x), ..., P_m^*(x)$.
7. Give the second derivatives of $(P'')_0^*(x), (P'')_1^*(x), ..., (P'')_m^*(x)$.
8. Give $x_1, x_2, ..., x_{m-1}$ the roots of shifted Legendre polynomial $P_{m-1}^*(x)$.
9. Give the exact solution to the problem $u_{ex}(x,t)$.

2: **For each time step:**

1. Compute the mass matrix A^{j+1}, the stiffness matrix B^{j+1} and the source term F_{j+1}.
2. Calculate the matrix $D^{j+1} = A^{j+1} - \tau B^{j+1}$ and the right-hand side $G^{j+1} = A^{j+1}U_{j+1} + \tau F_{j+1}$.
3. Solve the system $D^{j+1}U_{j+1} = G_{j+1}$.
4. We set $c_0 = U_{j+1}(1)$, $c_1 = U_{j+1}(2)$,$\ldots$, $c_m = U_{j+1}(m)$, and we set $u_m = \sum_{i=1}^{m} c_i P_i^*(x)$.
5. Draw in the same figure, the graphs of $u_m(x)$ and $u_{ex}(x, t_{j+1})$.
6. Calculate the L_2 norm: $\|u_m - u_{ex}\|_{L^2(0,1)}$ and the L^∞ norm: $\|u_m - u_{ex}\|_{L^\infty(0,1)}$.
7. Calculate the approximate reaction:
$$p(t_{j+1}) = \frac{1}{E(t_{j+1})}\left[(\beta-1)\sum_{i=0}^{m} c_i(t_{j+1})(P_i^*)'(0) + \int_0^1 S(x,t_{j+1})\,dx - \mathcal{D}_t^{(\alpha)}E(t_{j+1})\right].$$
8. Draw in the same figure, the graphs of $p(t_{j+1})$ and $p_{ex}(t_{j+1})$.
9. Calculate the norm: $|p(t_{j+1}) - p_{ex}(t_{j+1})|$.

Example 6.1 ([23]). We consider the following data:

$$\begin{cases} S(x,t) = 4\pi^2 \cos(2\pi x)\, e^{-4\pi^2 t} + 2t\,(1+\cos(2\pi x))\, e^{-4\pi^2 t + 10t^2}, \\ \varphi(x) = 1 + \cos(2\pi x), \\ E(t) = e^{-4\pi^2 t}, \\ \alpha = 1,\ \beta = 0. \end{cases}$$

For these data, the inverse time-dependent coefficient problem (4.1)- (4.4) is given by:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - p(t)\, u(x,t) + S(x,t), \\ u(x,0) = \varphi(x), \\ u(0,t) = u(1,t);\ u_x(1,t) = 0, \\ \int_0^1 u(x,t)\, dx = E(t). \end{cases} \tag{6.20}$$

This example was studied in [23] by different numerical method. In this case, the exact solution is given by:

$$u(x,t) = (1+\cos(2\pi x))\, e^{-4\pi^2 t};\ p(t) = 4\pi^2 + 2te^{10t^2}.$$

We apply Algorithm 1 for $m = 8$ with the numerical solution defined by:

$$u_8(x) = \sum_{i=1}^{8} c_i(t)\, P_i^*(x).$$

Table 6.1 compares error and relative error obtained by our algorithm and with the numerical method studied in [23]. In Table 6.2, we compare the L^2 errors at different values of T with $\tau = 0.0025$ and $h = 0.005$ with the results obtained by the method studied in [23]. From this comparative study, we can conclude that the numerical solutions obtained by our algorithm are very good. the curves of the exact and numerical solutions of $p(t)$ and $u(x,t)$ are given by Figure 6.1 and Figure 6.2.

Table 6.1: Error and Relative error of $p(t)$ with $\tau = 0,0455$, $h = 0,005$.

	Our Method		M. Ismailov and F. Kansa [23]	
Times	Error	Relative error	Error	Relative error
0.0455	9.66E-13	2.44E-14	0.0650	0.0016
0.0909	9.52E-13	2.40E-14	0.0647	0.0016
0.1364	1.25E-12	3.12E-14	0.0606	0.0015
0.1818	1.63E-12	4.09E-14	0.0588	0.0014
0.2273	1.76E-12	4.38E-14	0.0576	0.0014
0.2727	2.10E-12	5.18E-14	0.0561	0.0013
0.3182	2.27E-12	5.51E-14	0.0544	0.0013
0.3636	4.82E-12	1.14E-14	0.0524	0.0012
0.4091	1.58E-12	3.61E-14	0.0503	0.0011
0.4545	2.29E-11	4.90E-13	0.0485	0.0010
0.5000	3.64E-11	7.04E-13	0.0480	0.0009

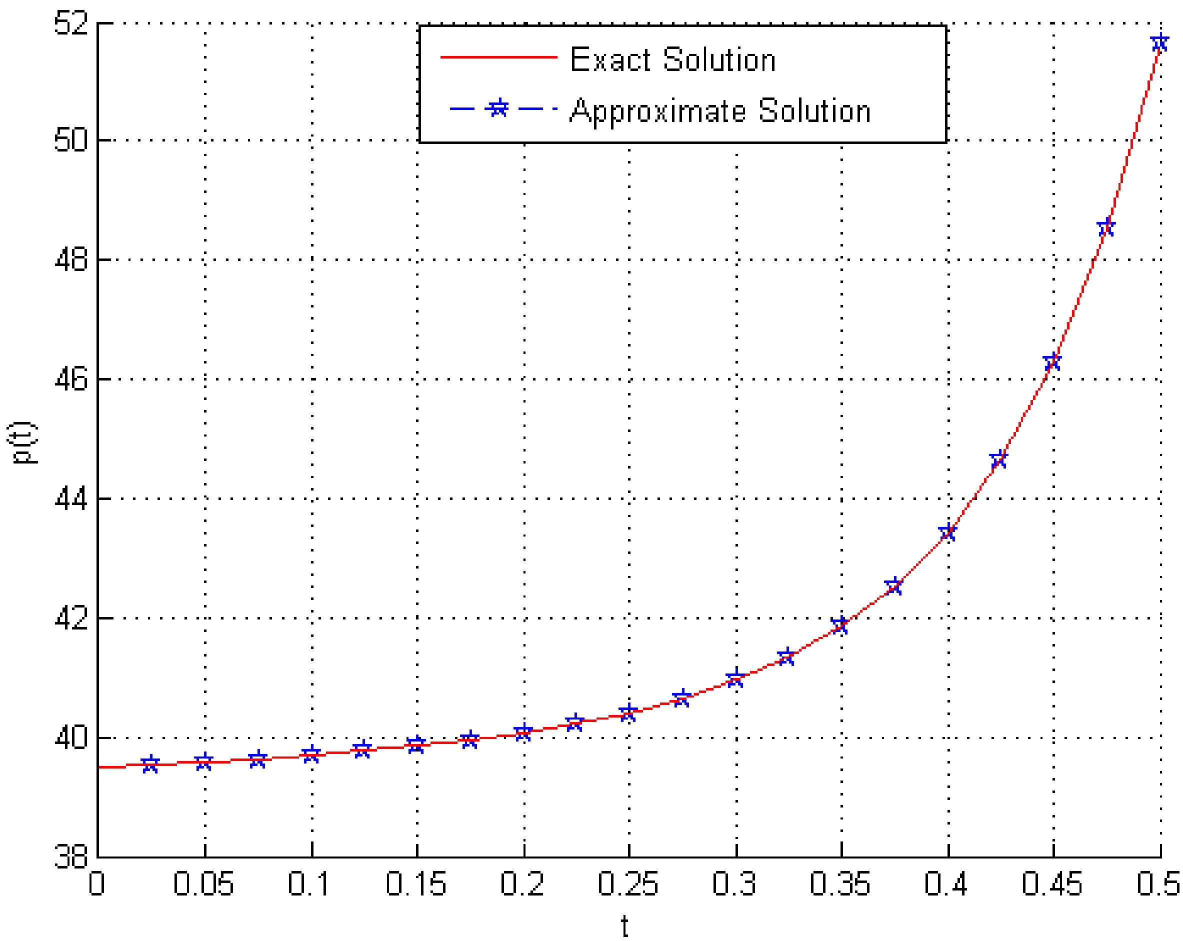

Figure 6.1: Exact and approximate of $p(t)$

Table 6.2: Error of $u(x,t)$ with $T = 0.5$, $\tau = 0,00025$, $h = 0,0005$.

	Our Method	M. Ismailov and F. Kansa [23]
Times	Error	Error
0.000250	0.00009	0.0079
0.050250	0.00013	0.0057
0.100250	0.00019	0.0034
0.150250	0.00024	0.0020
0.200250	0.00029	0.0014
0.250250	0.00034	0.0053
0.300250	0.00039	0.0062
0.350250	0.00044	0.0004
0.400250	0.00049	0.0080
0.450250	0.00054	0.0084

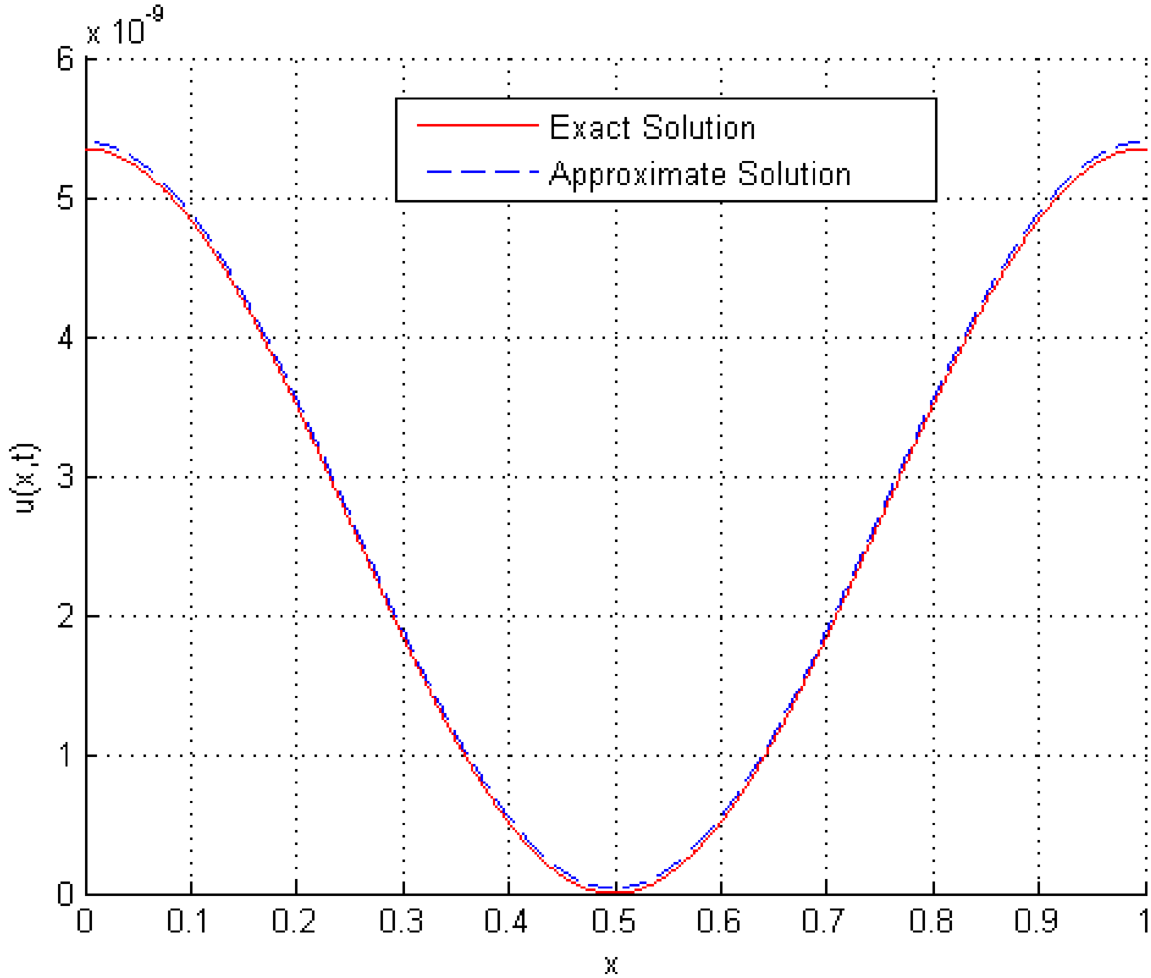

Figure 6.2: Exact and approximate of $u(x,t)$

Example 6.2. We consider the following data:

$$\begin{cases} T = 20, r\left(t\right) = e^{t^{\alpha}}, \varphi\left(x\right) = \left(x - x^2\right)^2, E\left(t\right) = e^{t^{\alpha}}/30, \\ S\left(x,t\right) = \left(e^{t^{\alpha}} + \alpha\right) e^{t^{\alpha}} \left(x - x^2\right)^2 - 2\left(1 - 6x + 6x^2\right) e^{t^{\alpha}}, \\ \text{Exact solution is } u_e = e^{t^{\alpha}} \left(x - x^2\right)^2. \end{cases} \tag{6.21}$$

☞ For $\alpha = 0.5$, the exact and approximate solution of $u\left(x,t\right)$ and $p\left(t\right)$ are given by:

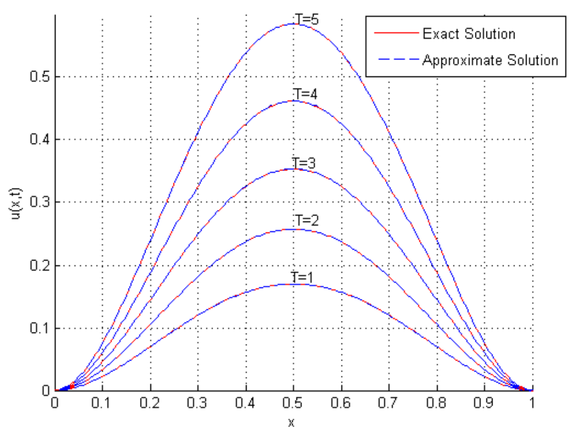

Figure 6.3: Exact and numerical solutions of $u\left(x,t\right)$.

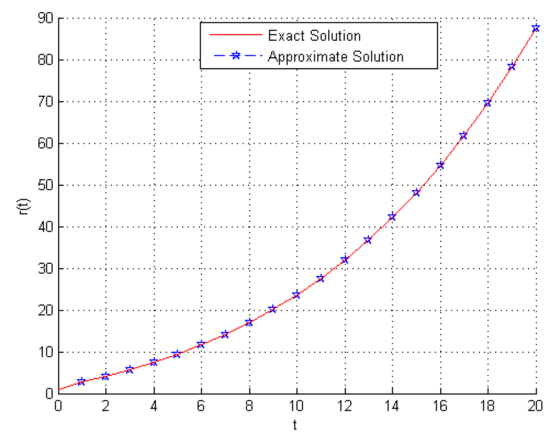

Figure 6.4: Exact and numerical solutions of $p\left(t\right)$.

☞ For $\alpha = 0.9$, the exact and approximate solution of $u\left(x,t\right)$ and $p\left(t\right)$ are given by:

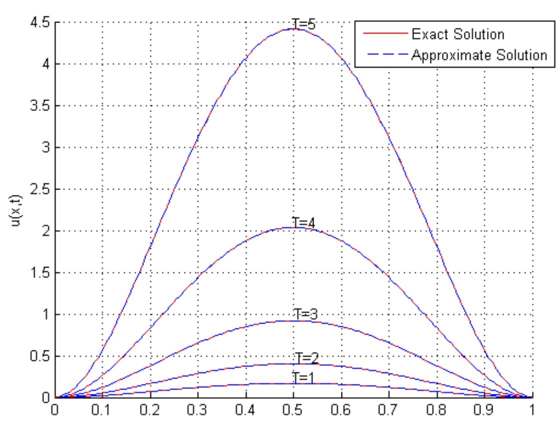

Figure 6.5: Exact and numerical solutions of $u\left(x,t\right)$.

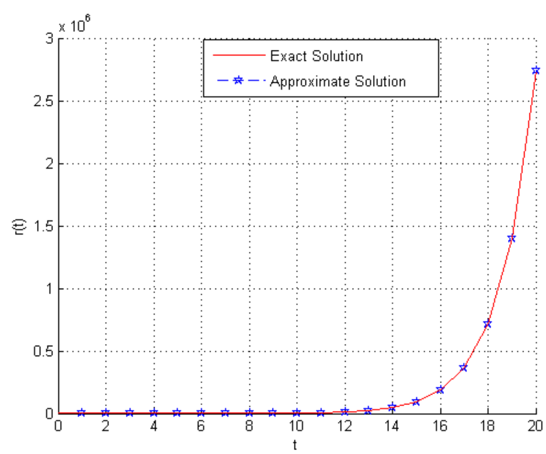

Figure 6.6: Exact and numerical solutions of $p\left(t\right)$.

☞ For $\alpha = 0.1$, the exact and approximate solution of $u(x,t)$ and $p(t)$ are given by:

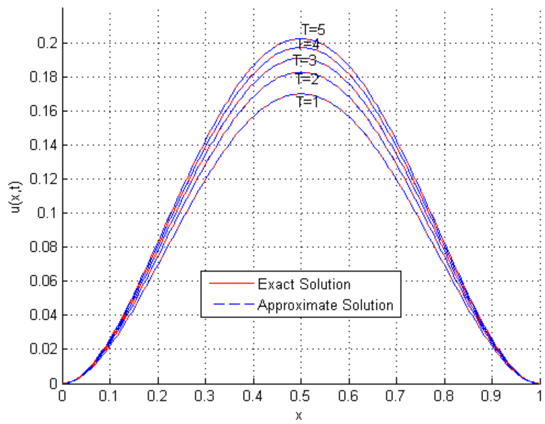

Figure 6.7: Exact and numerical solutions of $u(x,t)$.

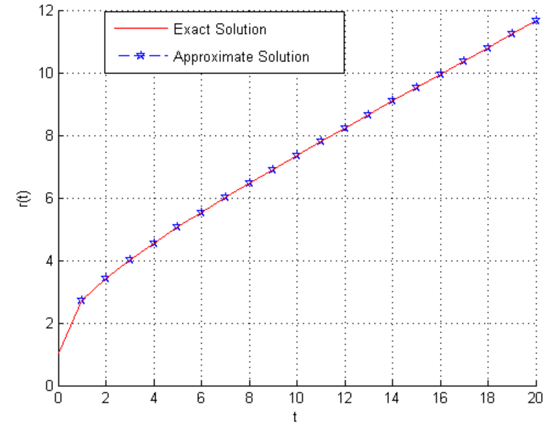

Figure 6.8: Exact and numerical solutions of $p(t)$.

Conclusion générale

In this thesis, we have studied various inverse problems related to time-fractional diffusion and reaction-diffusion equations with nonlocal and overdetermination conditions. Our main focus has been on the identification of time-dependent parameters—namely, source terms and coefficients—appearing in such models.

We first examined the inverse problem of determining a time-dependent source term in a time-fractional diffusion equation. A key challenge in this problem is the lack of completeness of the eigenfunction system. However, by employing the system of eigenfunctions together with their associated functions, which forms a basis in $L^2(0,1)$, we established well-posedness results. Under suitable assumptions on the input data, we proved the existence, uniqueness, and continuous dependence of the solution using a combination of the generalized Fourier method, Mittag-Leffler function estimates, and the Banach fixed-point theorem. The results of this study formed the basis of a research article published in an international journal [40].

Subsequently, we investigated an inverse problem aimed at identifying a time-dependent coefficient in a one-dimensional time-fractional reaction-diffusion equation. Here, the fractional derivative was considered in the conformable sense. Using Fourier analysis and Banach's contraction mapping principle, we showed that this problem also admits a unique and stable solution under appropriate conditions.

From a numerical standpoint, we proposed and analyzed a finite difference approximation scheme for the time-dependent source inverse problem. Furthermore, we developed an efficient numerical algorithm based on shifted Legendre polynomials. This approach transforms the inverse problem into a linear system of first-order differential equations, which is solved using the Backward Euler method. Several numerical examples were provided to validate the accuracy, robustness, and effectiveness of the proposed algorithm. Overall, this thesis contributes to both the theoretical analysis and numerical resolution of inverse problems involving time-fractional partial differential equations. The methodologies employed offer a solid framework for tackling a wide range of related problems in applied mathematics, physics, and engineering.

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