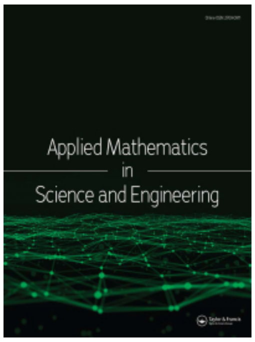

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Asymptotic behaviour of nonlinear viscoelastic wave equations with boundary feedback

Abdelbaki Choucha[a,b], Khaled Mahdi[c], Salah Boulaaras[d], Rashid Jan[e,f] and Taha Radwan[g]

[a]Department of Material Sciences, Faculty of Sciences, Amar Teleji Laghouat University, Laghouat, Algeria; [b]Laboratory of Mathematics and Applied Sciences, Ghardaia University, Laghouat, Algeria; [c]Department of Physics, Faculty of Sciences, University of M'Sila, M'Sila, Algeria; [d]Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia; [e]Institute of Energy Infrastructure (IEI), Department of Civil Engineering, College of Engineering, Universiti Tenaga Nasional (UNITEN), Kajang, Selangor, Malaysia; [f]Mathematics Research Center, Near East University, Nicosia, Turkey; [g]Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah, Saudi Arabia

ABSTRACT

In this study, we investigate a nonlinear viscoelastic wave equation subject to acoustic boundary conditions and a nonlinear distributed delay feedback acting on the boundary. The asymptotic behaviour of solutions is analysed by considering a general kernel formulation and imposing suitable assumptions.

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1. Introduction

The study of the asymptotic behaviour of a viscoelastic wave equation with nonlinear distributed delay and acoustic boundary conditions is fundamental to advancing our understanding of complex dynamical systems [1–8]. Several studies have investigated the asymptotic behaviour of viscoelastic wave equations with boundary feedback. Al-Mahdi and Al-Gharabli [9] analysed the energy decay properties of a viscoelastic equation incorporating past history and boundary feedback, providing conditions for stability. Messaoudi and Al-Gharabli [10] established a general decay result for a similar model, demonstrating the influence of memory effects on energy dissipation. Further advancements were made by Al-Gharabli et al. [11], who examined a viscoelastic system with nonlinear boundary feedback and a logarithmic source term, deriving decay estimates under suitable conditions. In a related study, the same authors [12] obtained general and optimal decay results for a viscoelastic equation with nonlinear boundary feedback, refining existing stability

CONTACT Taha Radwan t.radwan@qu.edu.sa Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah 51452, Saudi Arabia

criteria. These contributions provide a solid theoretical foundation for understanding the long-term behaviour of viscoelastic wave equations with boundary interactions.

This study addresses key theoretical challenges in the modelling of materials exhibiting both elastic and viscous properties, particularly within the framework of partial differential equations and control theory. By incorporating acoustic boundary feedback and nonlinear distributed delays, the analysis provides deeper insights into the stability and long-term behaviour of such systems. In addition to advancing the fundamental understanding of wave propagation in complex media, this work has significant implications for various scientific and engineering applications.

The below stated equation will be investigated in this work

$$\begin{cases} z_{tt} - \Delta z(t) + \displaystyle\int_0^t \mathscr{L}(t-\varkappa)\Delta z(\varkappa)\,\mathrm{d}\varkappa = 0, & \text{in } \mathfrak{A} \times \mathbb{R}_+, \\ \dfrac{\partial z}{\partial \upsilon} - \displaystyle\int_0^t \mathscr{L}(t-\varkappa)\dfrac{\partial}{\partial \upsilon} z(\varkappa)\,\mathrm{d}\varkappa + \mathscr{J}(z_t) = \mathscr{D}_t, & y \in \Lambda_0, t > 0, \\ z_t + \mathscr{I}(y)\mathscr{D}_t + \mathcal{N}(y)\mathscr{D} = 0, & y \in \Lambda_0, t > 0, \\ z(y,t) = 0, & \text{on } \Lambda_1 \times \mathbb{R}_+, \\ z(y,0) = z_0(y), \quad z_t(y,0) = z_1(y), & \text{in } \mathfrak{A}, \\ z_t(y,-t) = v_0(y,t), & \text{in } \Lambda_0 \times (0,\varphi_2), \\ \mathscr{D}(y,0) = \mathscr{D}_0(y), & y \in \Lambda_0, \end{cases} \tag{1}$$

where

$$\mathscr{J}(z_t) := \mathscr{U}_1 \mathscr{J}_1(z_t) + \int_{\varphi_1}^{\varphi_2} \mathscr{U}_2(j)\mathscr{J}_2(z_t(t-j))\,\mathrm{d}j. \tag{2}$$

In this study, we consider a bounded domain denoted by $\mathfrak{A} \subset \mathbb{R}^{\mathcal{M}}$ with $\mathcal{M} \geq 1$, possessing a smooth boundary $\partial\mathfrak{A} = \Lambda_1 \cup \Lambda_0$. Here, Λ_1 and Λ_0 are disjoint, closed subsets of $\partial\mathfrak{A}$. The outward unit normal vector to Λ is denoted by υ, while $\mathscr{U}_1$ represents a positive constant. Furthermore, we consider two non-negative constants φ_1 and φ_2 satisfying $\varphi_1 < \varphi_2$. The function $\mathscr{U}_2 : [\varphi_1, \varphi_2] \to \mathbb{R}$ characterizes the distributed time delay, while the memory kernel is represented by a positive function $\mathscr{L}$. Additionally, the functions $\mathscr{J}_1$ and $\mathscr{J}_2$ are introduced to describe specific aspects of the formulation.

Time delay has been recognized as a crucial factor in various physical and natural phenomena, as it directly affects the response to external forces, material transport processes, and the evolution of system states, all of which are inherently dependent on temporal dynamics. In recent years, the study of delay effects has gained significant attention in scientific research. Extensive investigations have been conducted on this form of damping due to its fundamental role in determining system stability and ensuring the existence of solutions.

In the absence of acoustic boundary conditions, where $\mathscr{J}1(\Xi) = \mathscr{J}2(\Xi) = \Xi$, this term can manifest in several forms: delay $(z_t(y, t-\varphi))$ as described in [13–16], time-varying delay $(z_t(y, t-\varphi(t)))$ as detailed in [17–20], and distributed delay $(\int_{\varphi_1}^{\varphi_2} \mathscr{U}(j) z_t(t-j)\,\mathrm{d}j)$ as explored in [21–24]. Numerous studies have investigated various issues related to the functions $\mathscr{J}1$ and $\mathscr{J}2$, particularly those involving delays within the equation or the boundary feedback.

The following problem has been investigated in [25]:

$$z_{tt} - \Delta z + \int_0^t h(t-j)\Delta z(j)\,\mathrm{d}j + \phi_1 g_1(z_t) + \phi_2 g_2(z_t(t-\varphi) = 0, \tag{3}$$

in which the authors focussed on the existence and stability of the global solution. In [26], the author considered the below

$$\begin{aligned} & z_{tt} - \Delta z = 0, \\ & z(y,t) = 0, \quad \text{on } \Lambda_0 \times \mathbb{R}_+, \\ & \frac{\mathrm{d}z}{\mathrm{d}\upsilon} + \phi_1 z_t + \phi_2 z_t(t-\varphi) = 0, \quad \text{on } \Lambda_1 \times \mathbb{R}_+, \end{aligned} \tag{4}$$

where the author proved that the general energy is exponentially stable under suitable supposition ($\phi_2 < \phi_1$). In [20], the the below stated problem is considered:

$$\begin{aligned} & z_{tt} - \Delta z + \int_0^t h(t-j)\Delta z(j)\,\mathrm{d}j + \phi_2 g_2(z_t(t-\varphi) = 0, \\ & z(y,t) = 0, \quad \text{on } \Lambda_0 \times \mathbb{R}_+, \\ & \frac{\mathrm{d}z}{\mathrm{d}\upsilon} + \phi_1 g_1(z_t) = 0, \quad \text{on } \Lambda_1 \times \mathbb{R}_+, \end{aligned} \tag{5}$$

where the authors established results for global existence and asymptomatic behaviour of the problem (5). The idea of viscoelasticity has been studied in numerous research work, including ([27–31]).

Conversely, the acoustic boundary conditions introduced by Ingard and Morse in their work [32] have been widely adopted in various problems, contributing to substantial advancements. Several researchers have extended their work, leading to a wide range of significant findings. For a more comprehensive understanding, we refer readers to the following studies: [13,17,32,33]. Additionally, for problems related to boundary dissipations, we recommend consulting [23,34,35]. In a recent study, the authors in [23] investigated problem (1) without considering acoustic boundary conditions, focussing on the general decay behaviour, particularly in the presence of a general kernel. Building upon and extending these findings, our objective is to analyse a specific problem that incorporates a nonlinear distributed delay within the boundary feedback while integrating acoustic boundary conditions. This key distinction differentiates our work from previous studies. By imposing appropriate assumptions on the involved functions, we will establish results concerning the asymptotic behaviour of the solutions.

Our work is organized as follows: The lemmas, concepts, and assumptions necessary for investigating the problem are presented in Section 2. Section 3 outlines the main findings of the study, while the general conclusion is provided in Section 4.

2. Preliminaries

In this section, we will presents the below stated assumptions for $\mathscr{U}_2, \mathscr{I}, \mathcal{N}, \mathscr{L}, \mathscr{J}_1$ and $\mathscr{J}_2$:

(A1) $\mathscr{L}:\mathbb{R}_+\to\mathbb{R}_+$ is a non-increasing $\mathcal{S}^1$ functions, which satisfies

$$\mathscr{L}(0)>0,\quad \zeta_0=\int_0^\infty \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa<\infty,\quad 1-\zeta_0=\zeta>0. \tag{6}$$

(A2) Assume a non-increasing $\mathcal{S}^1$ function $\exists\vartheta:\mathbb{R}_+\to\mathbb{R}_+$ which satisfies

$$\vartheta(t)\mathscr{L}(t)+\mathscr{L}'(t)\le 0,\quad \forall t\ge 0. \tag{7}$$

(A3) Take a convex and increasing function $P:\mathbb{R}_+\to\mathbb{R}_+$ and a non-decreasing $\mathcal{S}^1$ function $\mathscr{J}_1:\mathbb{R}\to\mathbb{R}$ and $P:\mathbb{R}_+\to\mathbb{R}_+$ with $\mathcal{S}^1(\mathbb{R}_+)\cap\mathcal{S}^1(]0,\infty[)$ which satisfies

$$\begin{aligned}&P(0)=0,\\ &P\quad\text{is linear on }[0,\omega],\quad\text{or}\quad P'(0)=0\quad\text{and}\quad P''(t)>0\quad\text{on }]0,\omega],\end{aligned} \tag{8}$$

and

$$\begin{aligned}&r_0|j|\le\mathscr{J}_1(j)\le r_1|j|\quad\text{if }|j|\ge\omega,\\ &j^2+\mathscr{J}_1^2(j)\le P^{-1}(j\mathscr{J}_1(j))\quad\text{if }|j|\le\omega,\end{aligned} \tag{9}$$

in which P^{-1} indicates the inverse of P function and ω,r_0,r_1 represents positive constants.

(A4) Let us take an odd non-decreasing $\mathcal{S}^1$ function $\mathscr{J}_2:\mathbb{R}\to\mathbb{R}$ with $\exists\xi_1>\frac{1}{2}$ and $r_2,\xi_2>0$,

$$\begin{aligned}&|\mathscr{J}_2'(j)|\le r_2,\\ &\xi_1 j\mathscr{J}_2(j)\le\mathscr{Q}(j)\le\xi_2 j\mathscr{J}_1(j),\end{aligned} \tag{10}$$

where $\mathscr{Q}(j)=\int_0^j\mathscr{J}_2(\varsigma)\,\mathrm{d}\varsigma$.

(A5) $\mathscr{U}_2:[\varphi_1,\varphi_2]\to\mathbb{R}$ is a bounded function, which holds

$$2\xi_2\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\,\mathrm{d}j<\mathscr{U}_1. \tag{11}$$

(A6)

$$\begin{aligned}&\exists\mathscr{I}_\jmath,\mathcal{N}_\jmath>0(\jmath=0,1),\quad\text{with }\mathscr{I}_0\le\mathscr{I}(y)\le\mathscr{I}_1,\\ &\mathcal{N}_0\le\mathcal{N}(y)\le\mathcal{N}_1,\quad\forall y\in\Lambda_0.\end{aligned} \tag{12}$$

Let

$$(\mathscr{L}\circ\kappa)(t):=\int_\Gamma\int_0^t\mathscr{L}(t-\varkappa)|\kappa(t)-\kappa(\varkappa)|^2\,\mathrm{d}\varkappa\,\mathrm{d}y.$$

Considering the following variables, as stated in [24]

$$\eta(y,\nu,j,t)=z_t(y,t-j\nu),\quad (y,\nu,j,t)\in\mathcal{D}=\Lambda_1\times(0,1)\times(\varphi_1,\varphi_2)\times\mathbb{R}_+$$

which satisfy

$$\begin{cases}j\eta_t(y,\nu,j,t)+\eta_\nu(y,\nu,j,t)=0,\\ \eta(y,0,j,t)=z_t(y,t).\end{cases} \tag{13}$$

Problem (1) can also be stated as

$$\begin{cases} z_{tt} - \Delta z(t) + \int_0^t \mathscr{L}(t-\varkappa)\Delta z(\varkappa)\,\mathrm{d}\varkappa = 0, & \text{in } \mathfrak{A} \times \mathbb{R}_+, \\ \dfrac{\partial z}{\partial v} - \int_0^t \mathscr{L}(t-\varkappa)\dfrac{\partial}{\partial v}z(\varkappa)\,\mathrm{d}\varkappa + \mathscr{U}_1 \mathscr{J}_1(z_t) & \\ \qquad + \int_{\varphi_1}^{\varphi_2} \mathscr{U}_2(j)\,\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j = \mathscr{D}_t, & y \in \Lambda_0, t > 0, \\ z_t + \mathscr{I}(y)\mathscr{D}_t + \mathcal{N}(y)\mathscr{D} = 0, & y \in \Lambda_0, t > 0, \\ j\eta_t(y,\nu,j,t) + \eta_\nu(y,\nu,j,t) = 0, & \text{on } \mathcal{D}, \\ z(y,t) = 0, & \text{on } \Lambda_1 \times \mathbb{R}_+, \\ z(y,0) = z_0(y), \quad z_t(y,0) = z_1(y), & \text{in } \mathfrak{A}, \\ \eta(y,\nu,j,0) = \nu_0(y,\nu j), & \text{in } \Lambda_0 \times (0,\varphi_2), \\ \mathscr{D}(y,0) = \mathscr{D}_0(y), & y \in \Lambda_0, \end{cases} \tag{14}$$

The energy functional is provided below.

Lemma 2.1: *Energy functional* E*, stated as*

$$\begin{aligned} \mathrm{E}(t) &= \frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa\right)\|\nabla z(t)\|_2^2 + \frac{1}{2}\int_{\Lambda_0} \mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda \\ &\quad + \frac{1}{2}(\mathscr{L} \circ \nabla z)(t) + \int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} j|\mathscr{U}_2(j)|\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda, \end{aligned} \tag{15}$$

satisfies

$$\begin{aligned} \mathrm{E}'(t) &\le -\phi_1 \int_{\Lambda_0} z_t \mathscr{J}_1(z_t)\,\mathrm{d}\Lambda + \frac{1}{2}(\mathscr{L}' \circ \nabla z)(t) \\ &\quad - \int_{\Lambda_0} \mathscr{I}(y)\mathscr{D}_t^2\,\mathrm{d}\Lambda - \frac{1}{2}\mathscr{L}(t)\|\nabla z(t)\|_2^2 \\ &\quad - \phi_2 \int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda \le 0, \end{aligned} \tag{16}$$

where $\phi_2 = 2\xi_1 - 1 > 0$ *with* $\phi_1 = \mathscr{U}_1 - 2\xi_2 \int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\,\mathrm{d}j > 0$.

Proof: From (14) through mathematical skills, we have the following

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|z_t(t)\|_2^2 + (1 - \int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa)\|\nabla z(t)\|_2^2 + (\mathscr{L} \circ \nabla z)(t) + \int_{\Lambda_0} \mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda\right\} \\ &\quad + \mathscr{U}_1 \int_{\Lambda_0} z_t \mathscr{J}_1(z_t)\,\mathrm{d}\Lambda - \frac{1}{2}(\mathscr{L}' \circ \nabla z)(t) + \frac{1}{2}\mathscr{L}(t)\|\nabla z(t)\|_2^2 \\ &\quad + \int_{\Lambda_0} \mathscr{I}(y)\mathscr{D}_t^2\,\mathrm{d}\Lambda + \int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\mathscr{J}_2(\eta(y,1,j,t))z_t\,\mathrm{d}j\,\mathrm{d}\Lambda = 0. \end{aligned} \tag{17}$$

Next, multiplying (14) by $|\mathscr{U}_2(j)|\mathscr{J}_2(\eta(y,\nu,j,t))$, and integrating over $\Lambda_0 \times (0,1) \times (\varphi_1,\varphi_2)$, and using $(13)_2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} j|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda$$

$$
\begin{aligned}
&= -\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|.\mathscr{J}_2(\eta)\eta_\nu \,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda \\
&= -\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\frac{\mathrm{d}}{\mathrm{d}\nu}\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda \\
&= \int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\left(\mathscr{Q}(\eta(y,0,j,t)) - \mathscr{Q}(\eta(y,1,j,t))\right)\,\mathrm{d}j\,\mathrm{d}\Lambda \\
&= \left(\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\,\mathrm{d}j\right)\int_{\Lambda_0}\mathscr{Q}(z_t)\,\mathrm{d}\Lambda \\
&\quad - \int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda. \qquad (18)
\end{aligned}
$$

Combining (17) and (18), we have (16) and the below

$$
\begin{aligned}
\mathrm{E}'(t) = &-\mathscr{U}_1\int_{\Lambda_0} z_t\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda + \frac{1}{2}(\mathscr{L}'\circ\nabla z)(t) - \frac{1}{2}\mathscr{L}(t)\|\nabla z(t)\|_2^2 \\
&+ \int_{\Lambda_0}\mathscr{I}(y)\mathscr{D}_t^2\,\mathrm{d}\Lambda - \int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\mathscr{J}_2(\eta(y,1,j,t))z_t\,\mathrm{d}j\,\mathrm{d}\Lambda \\
&+ \left(\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\,\mathrm{d}j\right)\int_{\Lambda_0}\mathscr{Q}(z_t)\,\mathrm{d}\Lambda \\
&- \int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda. \qquad (19)
\end{aligned}
$$

Let $\mathscr{Q}^*$ denote the conjugate function of the convex function $\mathscr{Q}$, which is given by

$$\mathscr{Q}^*(\varrho) = \sup_{t\geq 0}(\varrho t - \mathscr{Q}(t)).$$

Here, $\mathscr{Q}^*$ is the Legendre transform of $\mathscr{Q}$, see (Arnold [36, p. 61–62]):

$$\mathscr{Q}^*(\varrho) = \varrho(\mathscr{Q}')^{-1}(\varrho) - \mathscr{Q}[(\mathscr{Q}')^{-1}(\varrho)], \quad \forall \varrho \geq 0, \qquad (20)$$

and $\mathscr{Q}^*$ also satisfies the generalized Young inequality as follows:

$$\varrho t \leq \mathscr{Q}^*(\varrho) + \mathscr{Q}(t), \quad \forall \varrho, t \geq 0. \qquad (21)$$

Now, through $\mathscr{Q}$, we get

$$\mathscr{Q}^*(\varrho) = \varrho\mathscr{J}_2^{-1}(\varrho) - \mathscr{Q}(\mathscr{J}_2^{-1}(\varrho)), \quad \forall \varrho \geq 0. \qquad (22)$$

Hence, by (22) and (10), we have

$$
\begin{aligned}
\mathscr{Q}^*(\mathscr{J}_2(\eta(y,1,j,t)) &= \eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t)) - \mathscr{Q}(\eta(y,1,j,t)) \\
&\leq (1-\xi_1)\eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t)). \qquad (23)
\end{aligned}
$$

Applying (21), (23) and (10), we have

$$\mathrm{E}'(t) \leq -\mathscr{U}_1\int_{\Lambda_0} z_t\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda + \frac{1}{2}(\mathscr{L}'\circ\nabla z)(t) - \frac{1}{2}\mathscr{L}(t)\|\nabla z(t)\|_2^2$$

$$
\begin{aligned}
&+ (1-\xi_1)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda \\
&- \int_{\Lambda_0}\mathscr{I}(y)\mathscr{D}_t^2\,\mathrm{d}\Lambda + 2\left(\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\,\mathrm{d}j\right)\int_{\Lambda_0}\mathscr{Q}(z_t)\,\mathrm{d}\Lambda \\
&- \int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda, \\
\leq &- \left(\mathscr{U}_1 - 2\xi_2\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\,\mathrm{d}j\right)\int_{\Lambda_0} z_t\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda - \int_{\Lambda_0}\mathscr{I}(y)\mathscr{D}_t^2\,\mathrm{d}\Lambda \\
&+ (1-2\xi_1)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda \\
&+ \frac{1}{2}(\mathscr{L}'\circ\nabla z)(t) - \frac{1}{2}\mathscr{L}(t)\|\nabla z(t)\|_2^2. \qquad (24)
\end{aligned}
$$

By setting $\phi_1 = \mathscr{U}_1 - 2\xi_2\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\,\mathrm{d}j$ and $\phi_2 = 2\xi_1 - 1$, we achieve (16). Consequently, we get that E is a non-increasing function by the relations (6)–(11). ■

In the next step, using the Faedo-Galerkin approach and combining the results from [37–39], we prove the following result.

Theorem 2.2: *Assume* (6)–(11) *holds. Then, there exists a weak solution* $(z,\eta,\mathscr{D})$ *of problem* (14), *for any* $z_0, z_1 \in P^1_{\Lambda_0}(\mathfrak{A})\cap\mathscr{B}^2(\Gamma)$, $v_0 \in \mathscr{B}^2(\Lambda_0\times(0,1)\times(\varphi_1,\varphi_2))$ *and* $\mathscr{D}_0 \in \mathscr{B}^2(\Lambda_0)$, *with*

$$
\begin{aligned}
&z, z_t \in \mathcal{S}(]0,\mathscr{Y}[, P^1_{\Lambda_0}(\mathfrak{A}))\cap\mathcal{S}^1(]0,\mathscr{Y}[,\mathscr{B}^2(\mathfrak{A})),\\
&z_{tt} \in \mathcal{S}(]0,\mathscr{Y}[,\mathscr{B}^2(\mathfrak{A})),\\
&\eta \in \mathcal{S}(]0,\mathscr{Y}[,\mathscr{B}^2(\Lambda_0\times(0,1)\times(\varphi_1\times\varphi_2))),\\
&\mathscr{D},\mathscr{D}_t \in \mathscr{B}^2(\mathbb{R}_+,\mathscr{B}^2(\Lambda_0)).
\end{aligned}
$$

The below stated Lemma will be utilized, in order to get the result.

Lemma 2.3 (Jensen's inequality): *If P represents a convex function on* $[a,a_1]$, $\mathscr{L}:\Sigma\to[a,a_1]$ *and take an integrable function i on* Σ *such that* $i(y)\geq 0$ *and* $\int_\Sigma i(y)\,\mathrm{d}y = \mathcal{L} > 0$, *then*

$$
P\left(\frac{1}{\mathcal{L}}\int_\Sigma \mathscr{L}(y)i(y)\,\mathrm{d}y\right) \leq \frac{1}{\mathcal{L}}\int_\Sigma P(\mathscr{L}(y))i(y)\,\mathrm{d}y. \qquad (25)
$$

3. General decay

The proof of the general decay result of the system (14) will be presented in this section. We will proceed in the following manner

$$
\Omega(t) := \int_{\mathfrak{A}} z(t)z_t(t)\,\mathrm{d}y + \int_{\Lambda_0} z\mathscr{D}\,\mathrm{d}\Lambda + \frac{1}{2}\int_{\Lambda_0}\mathscr{I}(y)\mathscr{D}^2\,\mathrm{d}\Lambda, \qquad (26)
$$

$$\Xi(t) := -\int_{\mathfrak{A}} z_t \int_0^t \mathscr{L}(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\,\mathrm{d}y, \tag{27}$$

and

$$\Upsilon(t) := \int_{\Lambda_1}\int_0^1\int_{\varphi_1}^{\varphi_2} je^{-\nu j}|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda. \tag{28}$$

Lemma 3.1: *Functional $\Omega(t)$ stated in (26) holds, for any $\omega_1,\omega_2,\omega_3 > 0$*

$$\begin{aligned}\Omega'(t) &\le -(k-2\omega_1-(\omega_2+\omega_3)r_p)\|\nabla z\|_2^2 + \|z_t\|_2^2 - \int_{\Lambda_0}\mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda\\ &\quad + r(\omega_1)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda + r(\omega_1)\int_{\Lambda_0}\mathscr{D}_t^2\,\mathrm{d}\Lambda + r(\omega_2)(\mathscr{L}\circ\nabla z)(t)\\ &\quad + r(\omega_3)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)\,\mathscr{J}_2^2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda.\end{aligned} \tag{29}$$

Proof: From (26) and $(14)_{1,3}$, we have the following

$$\begin{aligned}\Omega'(t) &= \|z_t\|_2^2 + \int_{\mathfrak{A}} z_{tt}z\,\mathrm{d}y + \int_{\Lambda_0} z\mathscr{D}_t\,\mathrm{d}\Lambda + \int_{\Lambda_0} z_t\mathscr{D}\,\mathrm{d}\Lambda + \int_{\Lambda_0}\mathscr{I}(y)\mathscr{D}_t\mathscr{D}\,\mathrm{d}\Lambda\\ &= \|z_t\|_2^2 - \left(1-\int_0^t\mathscr{L}(\varkappa)\,\mathrm{d}\varkappa\right)\|\nabla z\|_2^2 - \int_{\Lambda_0}\mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda + \underbrace{2\int_{\Lambda_0} z\mathscr{D}_t\,\mathrm{d}\Lambda}_{I_0}\\ &\quad + \underbrace{\int_{\mathfrak{A}}\nabla z(t)\int_0^t\mathscr{L}(t-\varkappa)(\nabla z(t)-\nabla z(\varkappa))\,\mathrm{d}\varkappa\,\mathrm{d}y}_{I_1}\\ &\quad \underbrace{-\,\mathscr{U}_1\int_{\Lambda_0} z\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda}_{I_2} + \underbrace{\int_{\Lambda_0} z\int_{\varphi_1}^{\varphi_2}\left(\mathscr{U}_2(j)\,\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\right)\,\mathrm{d}\Lambda}_{I_3}.\end{aligned} \tag{30}$$

We estimate the last four terms on the right-hand side of (30). By applying Young's, Hölder's, and Poincar'e's inequalities, along with (6), for $\omega_\jmath > 0,\ \jmath = 1,2,3$, we have

$$\begin{aligned} I_0 &\le \omega_1\|\nabla z\|_2^2 + r(\omega_1)\int_{\Lambda_0}\mathscr{D}_t^2\,\mathrm{d}\Lambda,\\ I_1 &\le \omega_1\|\nabla z\|_2^2 + r(\omega_1)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda,\end{aligned} \tag{31}$$

and

$$I_2 \le \omega_2 r_p\|\nabla z\|_2^2 + r(\omega_2)(\mathscr{L}\circ\nabla z)(t). \tag{32}$$

In similar manner, we achieve

$$I_3 \le \omega_3 r_p\|\nabla z\|_2^2 + r(\omega_3)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)\,\mathscr{J}_2^2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda. \tag{33}$$

Combining (31)–(33) and (30), we get (29). ■

Lemma 3.2: *The functional* $\Xi(t)$ *stated in* (27) *holds, for any* $\rho_1, \rho_2 > 0$ *as*

$$\Xi'(t) \leq -\left(\int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa - \rho_2\right)\|z_t\|_2^2 + \rho_1(2-k)\|\nabla z\|_2^2 - r(\rho_2)(\mathscr{L}' \circ \nabla z)(t)$$
$$+ r(\rho_1)(\mathscr{L} \circ \nabla z)(t) + r\int_{\Lambda_0} \mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda + r\int_{\Lambda_0} \mathscr{D}_t^2\,\mathrm{d}\Lambda$$
$$+ r\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)\,\mathscr{J}_2^2(\varkappa(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda. \tag{34}$$

Proof: From (27) and (14), we have the following

$$\Xi'(t) = -\int_{\mathfrak{A}} z_{tt}\int_0^t \mathscr{L}(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\,\mathrm{d}y$$
$$-\int_{\mathfrak{A}} z_t\int_0^t \mathscr{L}'(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\,\mathrm{d}y - \left(\int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa\right)\|z_t\|_2^2$$
$$= \int_{\mathfrak{A}}\left[\Delta z - \int_0^t \mathscr{L}(t-\varkappa)\Delta z(\varkappa))\,\mathrm{d}\varkappa\right]\left[\int_0^t \mathscr{L}(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\right]\mathrm{d}y$$
$$-\int_{\mathfrak{A}} z_t\int_0^t \mathscr{L}'(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\,\mathrm{d}y - \left(\int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa\right)\|z_t\|_2^2$$
$$= \underbrace{\int_{\mathfrak{A}} \nabla z\int_0^t \mathscr{L}(t-\varkappa)(\nabla z(t)-\nabla z(\varkappa))\,\mathrm{d}\varkappa\,\mathrm{d}y}_{J_1}$$
$$\underbrace{-\int_{\mathfrak{A}}\left(\int_0^t \mathscr{L}(t-\varkappa)\nabla z(\varkappa)\,\mathrm{d}\varkappa\right).\left(\int_0^t \mathscr{L}(t-\varkappa)(\nabla z(t)-\nabla z(\varkappa))\,\mathrm{d}\varkappa\right)\mathrm{d}y}_{J_2}$$
$$\underbrace{-\mathscr{U}_1\int_{\Lambda_0} \mathscr{J}_1(z_t)\left(\int_0^t \mathscr{L}(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\right)\mathrm{d}\Lambda}_{J_3}$$
$$\underbrace{-\int_{\Lambda_0}\left(\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\,\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\right)\left(\int_0^t \mathscr{L}(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\right)\mathrm{d}\Lambda}_{J_4}$$
$$\underbrace{-\int_{\mathfrak{A}} z_t\int_0^t \mathscr{L}'(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\,\mathrm{d}y}_{J_5} - \left(\int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa\right)\|z_t\|_2^2$$
$$+\underbrace{\int_{\Lambda_0} \mathscr{D}_t\left(\int_0^t \mathscr{L}(t-\varkappa)(z(t)-z(\varkappa))\,\mathrm{d}\varkappa\right)\mathrm{d}\Lambda}_{J_6}. \tag{35}$$

From (35) and (6) through Young's, Poincaré's, and Hölder's inequalities, we have

$$J_1 \leq \rho_1\|\nabla z\|_2^2 + r(\rho_1)(\mathscr{L} \circ \nabla z)(t), \tag{36}$$

and

$$J_2 \leq \left(\int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa\right)\int_{\mathfrak{A}}\left(\nabla z(t)\int_0^t \mathscr{L}(t-\varkappa)(\nabla z(t)-\nabla z(\varkappa))\,\mathrm{d}\varkappa\right)\mathrm{d}y$$
$$-\int_{\mathfrak{A}}\left(\int_0^t \mathscr{L}(t-\varkappa)(\nabla z(t)-\nabla z(\varkappa))\,\mathrm{d}\varkappa\right)^2\mathrm{d}y$$
$$\leq \rho_1(1-k)\|\nabla z\|_2^2 + r(\rho_1)(\mathscr{L}\circ\nabla z)(t), \tag{37}$$

Similarly, we have

$$\begin{aligned}
J_3 &\leq r\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda + r(\mathscr{L}\circ\nabla z)(t),\\
J_4 &\leq r\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)\mathscr{J}_2^2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda + r(\mathscr{L}\circ\nabla z)(t),\\
J_5 &\leq \rho_2\|z_t\|_2^2 - r(\rho_2)(\mathscr{L}'\circ\nabla z)(t),\\
J_6 &\leq r\int_{\Lambda_0}\mathscr{D}_t^2\,\mathrm{d}\Lambda + r(\mathscr{L}\circ\nabla z)(t).
\end{aligned} \tag{38}$$

Putting (36)–(38) into (35), gives (34). ■

Lemma 3.3: *The functional $\Upsilon(t)$ given in* (28) *satisfies as*

$$\Upsilon'(t) \leq -\psi_1\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} j|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda + \mathscr{U}_1\int_{\Lambda_0} z_t\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda$$
$$-\psi_1\xi_1\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\eta\left(y,1,j,t\right)\mathscr{J}_2(\eta\left(y,1,j,t\right))\,\mathrm{d}j\,\mathrm{d}\Lambda. \tag{39}$$

Proof: From $\Upsilon(t)$ and (14) through mathematical skills, we have

$$\begin{aligned}
\Upsilon'(t) &= -\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} e^{-j\nu}|\mathscr{U}_2(j)|.\eta_\nu\mathscr{J}_2(\eta(\left(y,\nu,j,t\right))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda\\
&= -\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} je^{-j\nu}|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda\\
&\quad -\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\left[e^{-j}\mathscr{Q}(\eta\left(y,1,j,t\right)) - \mathscr{Q}(\eta\left(y,0,j,t\right))\right]\mathrm{d}j\,\mathrm{d}\Lambda.
\end{aligned}$$

Using $\eta(y,0,j,t) = z_t(y,t)$ and noting that $e^{-j} \leq e^{-j\nu} \leq 1$ for any $0 < \nu < 1$, and setting $\psi_1 = e^{-\varphi_2}$, we have

$$\Upsilon'(t) \leq -\psi_1\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} j|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda$$
$$-\psi_1\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\mathscr{Q}(\eta\left(y,1,j,t\right))\,\mathrm{d}j\,\mathrm{d}\Lambda$$

$$+\left(\int_{\varphi_1}^{\varphi_2} |\mathscr{U}_2(j)|\,\mathrm{d}j\right)\int_{\Lambda_0} \mathscr{Q}(z_t)\,\mathrm{d}\Lambda,$$

by (10) and (11), we found (39). ∎

For some positive constants $\mathcal{M}$, $\mathscr{P}$ which will be determined later, define the functional

$$\mathcal{W}(t) := \mathcal{M}\mathrm{E}(t) + \Omega(t) + \mathscr{P}\Xi(t) + \Upsilon(t). \tag{40}$$

Lemma 3.4: *There exists $\mathfrak{C}_\jmath, t_0 > 0,\ \jmath = 1,\dots,5$ which satisfies the following*

$$\mathcal{W}'(t) \le -\mathfrak{C}_1\mathrm{E}(t) + \mathfrak{C}_2\int_{\Lambda_0} \mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda + \mathfrak{C}_3(\mathscr{L}\circ\nabla z)(t), \quad \forall t \ge t_0. \tag{41}$$

and

$$\mathfrak{C}_4\mathrm{E}(t) \le \mathcal{W}(t) \le \mathfrak{C}_5\mathrm{E}(t). \tag{42}$$

Proof: Since the function $\mathscr{L}$ is continuous and positive for all $t_0 > 0$, we get

$$\int_0^t \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa \ge \int_0^{t_0} \mathscr{L}(\varkappa)\,\mathrm{d}\varkappa := \mathscr{L}_0, \quad \forall t \ge t_0.$$

By differentiating (40) and utilizing (16), along with Lemmas 3.1, 3.2, and 3.3, one have

$$\begin{aligned}
\mathcal{W}'(t) &:= \mathcal{M}\mathrm{E}'(t) + \Omega'(t) + \mathscr{P}\Xi'(t) + \Upsilon'(t)\\
&\le -\left(\mathscr{P}(\mathscr{L}_0-\rho_2)-1\right)\|z_t\|_2^2 - \int_{\Lambda_0}\mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda\\
&\quad -\left(\mathcal{M}\mathscr{I}_0 - r\mathscr{P} - r(\omega_1)\right)\int_{\Lambda_0}\mathscr{D}_t^2\,\mathrm{d}\Lambda\\
&\quad -\left((k-2\omega_1 - r_p(\omega_2+\omega_3)) - \mathscr{P}\rho_1(2-k)\right)\|\nabla z\|_2^2\\
&\quad +\left(r(\omega_2)+\mathscr{P}r(\rho_1)\right)(\mathscr{L}\circ\nabla z)(t) + \left(\frac{\mathcal{M}}{2} - \mathscr{P}r(\rho_2)\right)(\mathscr{L}'\circ\nabla z)(t)\\
&\quad +\left(r(\omega_1)+r\mathscr{P}\right)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda - (\phi_1\mathcal{M} - \mathscr{U}_1)\int_{\Lambda_0} z_t\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda\\
&\quad +\left(r(\omega_3)+\mathscr{P}r\right)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\mathscr{J}_2^2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda\\
&\quad -\psi_1\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} j|\mathscr{L}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda\\
&\quad -\left(\phi_2\mathcal{M}+\psi_1\xi_1\right)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{L}_2(j)\|\eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda.
\end{aligned} \tag{43}$$

Choosing $\omega_j, j = 1,2,3$ small enough that

$$k_1 := k - 2\omega_1 - r_p(\omega_2 + \omega_3)) > 0.$$

Let

$$\rho_2 = \frac{\mathscr{L}_0}{2},$$

now pick $\mathscr{P}$ large enough with

$$\frac{\mathscr{L}_0}{2}\mathscr{P} - 1 > 0,$$

then, we choose

$$\rho_1 = \frac{k_1}{2(2-k)\mathscr{P}}.$$

Hence, the above (43) becomes

$$\begin{aligned}
\mathcal{W}'(t) \le & -d_1\|z_t\|_2^2 - d_2\|\nabla z\|_2^2 + d_3(\mathscr{L}\circ\nabla z)(t) + \left(\frac{\mathcal{M}}{2} - r\right)(\mathscr{L}'\circ\nabla z)(t) \\
& + d_4\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda - (\phi_1\mathcal{M} - \mathscr{U}_1)\int_{\Lambda_0} z_t\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda - \int_{\Lambda_0}\mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda \\
& + d_5\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\mathscr{J}_2^2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda - (\mathcal{M}\mathscr{I}_0 - r)\int_{\Lambda_0}\mathscr{D}_t^2\,\mathrm{d}\Lambda \\
& - \psi_1\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} j|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda \\
& - (\phi_2\mathcal{M} + \psi_1\xi_1)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda. \qquad (44)
\end{aligned}$$

By using (10), we get

$$\begin{aligned}
\mathcal{W}'(t) \le & -d_1\|z_t\|_2^2 - d_2\|\nabla z\|_2^2 + d_3(\mathscr{L}\circ\nabla z)(t) + \left(\frac{\mathcal{M}}{2} - r\right)(\mathscr{L}'\circ\nabla z)(t) \\
& + d_4\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda - (\phi_1\mathcal{M} - \mathscr{U}_1)\int_{\Lambda_0} z_t\mathscr{J}_1(z_t)\,\mathrm{d}\Lambda - \int_{\Lambda_0}\mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda \\
& - \psi_1\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} j|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda - (\mathcal{M}\mathscr{I}_0 - r)\int_{\Lambda_0}\mathscr{D}_t^2\,\mathrm{d}\Lambda \\
& - (\phi_2\mathcal{M} + \psi_1\xi_1 - r_2 d_5)\int_{\Lambda_0}\int_{\varphi_1}^{\varphi_2}|\mathscr{U}_2(j)|\eta(y,1,j,t)\mathscr{J}_2(\eta(y,1,j,t))\,\mathrm{d}j\,\mathrm{d}\Lambda. \qquad (45)
\end{aligned}$$

From (26)–(28), and by applying Poincaré's, Hölder's, and Young's inequalities, we obtain

$$|\mathcal{W}(t) - \mathcal{M}\mathrm{E}(t)| \le \frac{1}{2}\|z_t(t)\|_2^2 + r_p\|\nabla z(t)\|_2^2 + \left(\frac{\mathscr{I}_1}{2\mathcal{N}_0} + \frac{1}{2\mathcal{N}_0}\right)\int_{\Lambda_0}\mathcal{N}(y)\mathscr{D}^2\,\mathrm{d}\Lambda$$

$$+\frac{\mathscr{P}}{2}\left(\|z_t(t)\|_2^2 + r_p(\mathscr{L}\circ\nabla z)(t)\right)$$
$$+\int_{\Lambda_0}\int_0^1\int_{\varphi_1}^{\varphi_2} je^{-\nu j}|\mathscr{U}_2(j)|.\mathscr{Q}(\eta(y,\nu,j,t))\,\mathrm{d}j\,\mathrm{d}\nu\,\mathrm{d}\Lambda. \tag{46}$$

Utilizing that $e^{-\nu j} < 1$, we get

$$|\mathcal{W}(t) - \mathcal{M}\mathrm{E}(t)| \leq \mathcal{S}\mathrm{E}(t). \tag{47}$$

Hence

$$(\mathcal{M}-\mathcal{S})\mathrm{E}(t) \leq \mathcal{W}(t) \leq (\mathcal{M}+\mathcal{S})\mathrm{E}(t). \tag{48}$$

Now, selecting $\mathcal{M}$ large enough with

$$\mathcal{M}-\mathcal{S} > 0, \quad \phi_1\mathcal{M} - \mathscr{U}_1 > 0, \quad \frac{\mathcal{M}}{2} - r > 0, \quad \phi_2\mathcal{M} + \psi_1\xi_1 - r_2 d_5 > 0,$$
$$\mathcal{M}\mathscr{I}_0 - r > 0.$$

Exploiting (15), estimates (45) and (48), respectively, gives (41) and (42). ■

Theorem 3.5: *Let* (6)–(11) *holds, then one can find constants* $\lambda_1 > 0, \lambda_2 > 0, t_0$ *and* $\omega_0 \in (0,\omega]$ *with energy of* (14) *fulfilling:*

$$\mathrm{E}(t) \leq \lambda_1 P^{-1}\left\{\lambda_2\left(1+\int_{t_0}^t \vartheta(\varsigma)\,\mathrm{d}\varsigma\right)\right\}, \quad \forall t \geq t_0, \tag{49}$$

where

$$P(t) := \int_t^1 \frac{1}{\mathcal{H}(\varrho)}\,\mathrm{d}\varrho, \tag{50}$$

with

$$\mathcal{H}(t) = \begin{cases} t, & \textit{if } P \quad \textit{is linear on } [0,\omega], \\ tP'(\omega_0 t), & \textit{if } P'(0) = 0 \quad \textit{and} \quad P'' > 0 \quad \textit{on } (0,\omega], \end{cases} \tag{51}$$

Proof: Multiplying $\vartheta(t)$ with (41), utilizing (6) and (16), we have

$$\begin{aligned}\vartheta(t)\mathcal{W}'(t) &\leq -\mathfrak{C}_1\vartheta(t)\mathrm{E}(t) + \mathfrak{C}_2\vartheta(t)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda + \mathfrak{C}_3\vartheta(t)(\mathscr{L}\circ\nabla z)(t)\\ &\leq -\mathfrak{C}_1\vartheta(t)\mathrm{E}(t) + \mathfrak{C}_2\vartheta(t)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda - \mathfrak{C}_3(\mathscr{L}'\circ\nabla z)(t)\\ &\leq -\mathfrak{C}_1\vartheta(t)\mathrm{E}(t) + \mathfrak{C}_2\vartheta(t)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda - 2\mathfrak{C}_3\mathrm{E}'(t).\end{aligned} \tag{52}$$

As, the function $\vartheta(t)$ non-increasing, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vartheta(t)\mathcal{W}(t)+2\mathfrak{C}_3\mathrm{E}(t)\right) \leq -\mathfrak{C}_1\vartheta(t)\mathrm{E}(t)+\mathfrak{C}_2\vartheta(t)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda. \tag{53}$$

Let

$$\mathscr{V}(t) := \vartheta(t)\mathcal{W}(t)+2\mathfrak{C}_3\mathrm{E}(t) \sim \mathrm{E}(t). \tag{54}$$

Therefore, we get

$$\mathscr{V}'(t) \leq -\mathfrak{C}_1\vartheta(t)\mathrm{E}(t)+\mathfrak{C}_2\vartheta(t)\int_{\Lambda_0}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda, \quad \forall t \geq t_0. \tag{55}$$

Now, the last term of the inequality (55) will be evaluated for our main results. For this, take the following

$$\Lambda_0^1 := \left\{y \in \Lambda_0 : \quad |z_t| > \omega\right\} \quad \text{and} \quad \Lambda_0^2 := \left\{y \in \Lambda_0 : \quad |z_t| \leq \omega\right\}. \tag{56}$$

By (9) and (16), we have

$$\mathfrak{C}_2\vartheta(t)\int_{\Lambda_0^1}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda \leq \mathfrak{C}_2\vartheta(0)\int_{\Lambda_0^1}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda \leq -\lambda_3\mathrm{E}'(t), \tag{57}$$

where $\lambda_3 = \frac{\mathfrak{C}_2 r_1\vartheta(0)}{\phi_1}$.

Now, the below two cases will be discussed.

Case 1: P is linear on $[0,\omega]$: According (9) and (16), we get

$$\mathfrak{C}_2\vartheta(t)\int_{\Lambda_0^2}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda \leq \mathfrak{C}_2\vartheta(0)\int_{\Lambda_0^2}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda \leq -\lambda_4\mathrm{E}'(t), \tag{58}$$

where $\lambda_4 = \mathfrak{C}_2 r\vartheta(0)/\phi_1$. Putting (57) and (58) into (55), and by $(51)_1$, we find

$$\begin{aligned}\mathscr{V}_1'(t) &\leq -\mathfrak{C}_1\vartheta(t)\mathrm{E}(t)\\ &= -\lambda_5\vartheta(t)\mathcal{H}\left(\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right), \quad \forall t \geq t_0,\end{aligned} \tag{59}$$

where

$$\mathscr{V}_1(t) = \left(\mathscr{V}(t)+\lambda\mathrm{E}(t)\right) \sim \mathrm{E}(t), \tag{60}$$

and $\lambda = \lambda_3+\lambda_4$, $\lambda_5 = \mathfrak{C}_1\mathrm{E}(0)$.

Integrating (59) over (t_0,t) and using (60), we get (49).

Case 2: P is nonlinear: From (9), (16) and by Jensen's inequality (25) with $\Sigma = \Lambda_0^2$, $i(y) = 1$ and $v(y) = P^{-1}(z_t(y)\mathscr{J}_1(z_t(y)))$, we get

$$\lambda_2\vartheta(t)\int_{\Lambda_0^2}\mathscr{J}_1^2(z_t)\,\mathrm{d}\Lambda \leq \lambda_2\vartheta(t)\int_{\Lambda_0^2}P^{-1}(z_t\mathscr{J}_1(z_t))\,\mathrm{d}\Lambda$$

$$\leq \lambda_2 \vartheta(t)|\Lambda_0^2|P^{-1}\left(\frac{1}{|\Lambda_0^2|}\int_{\Lambda_0^2} z_t \mathscr{J}_1(z_t)\,\mathrm{d}\Lambda\right)$$
$$\leq \lambda_2 \vartheta(t)|\Lambda_0^2|P^{-1}\left(-\frac{\mathrm{E}'(t)}{\phi_1|\Lambda_0^2|}\right), \tag{61}$$

Substituting (57) and (61) into (55), we find

$$\mathscr{V}_2'(t) \leq -\mathfrak{C}_1\vartheta(t)\mathrm{E}(t) + \lambda_6\vartheta(t)P^{-1}\left(-\frac{\mathrm{E}'(t)}{\phi_1|\Lambda_0^2|}\right), \quad \forall t \geq t_0, \tag{62}$$

where

$$\mathscr{V}_2(t) = (\mathscr{V}(t) + \lambda_3\mathrm{E}(t)) \sim \mathrm{E}(t), \tag{63}$$

and $\lambda_6 = \lambda_2|\Lambda_0^2|$.

Now, for $0 < \omega_0 < \omega$ and $\rho_0 > 0$, according (62), (20) and (21), we have

$$\begin{aligned}
&\left\{P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathscr{V}_2(t) + \rho_0\mathrm{E}(t)\right\}' \\
&= \omega_0\frac{\mathrm{E}'(t)}{\mathrm{E}(0)}P''\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathscr{V}_2(t) + P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathscr{V}_2'(t) + \rho_0\mathrm{E}'(t) \\
&\leq -\mathfrak{C}_1\vartheta(t)P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathrm{E}(t) + \lambda_6\vartheta(t)P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)P^{-1}\left(-\frac{\mathrm{E}'(t)}{\phi_1|\Lambda_0^2|}\right) + \rho_0\mathrm{E}'(t) \\
&\leq -\mathfrak{C}_1\vartheta(t)P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathrm{E}(t) + \lambda_6\vartheta(t)P^*\left(P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\right) \\
&\quad - \frac{\lambda_6\vartheta(t)}{\phi_1|\Lambda_0^2|}\mathrm{E}'(t) + \rho_0\mathrm{E}'(t) \\
&= -\mathfrak{C}_1\vartheta(t)P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathrm{E}(t) + \lambda_6\vartheta(t)P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)} \\
&\quad - \lambda_6\vartheta(t)P\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right) - \frac{\lambda_6\vartheta(t)}{\phi_1|\Lambda_0^2|}\mathrm{E}'(t) + \rho_0\mathrm{E}'(t) \\
&\leq -(\mathfrak{C}_1\mathrm{E}(0) - \lambda_6\vartheta_0)\vartheta(t)P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\frac{\mathrm{E}(t)}{\mathrm{E}(0)} + \left(\rho_0 - \frac{\lambda_6\vartheta(0)}{\phi_1|\Lambda_0^2|}\right)\mathrm{E}'(t).
\end{aligned} \tag{64}$$

Next, taking ω_0 in a way that

$$\lambda_7 := \mathfrak{C}_1\mathrm{E}(0) - \lambda_6\omega_0 > 0,$$

and select ρ_0 large such that

$$\rho_0 - \frac{\lambda_6\vartheta(0)}{\phi_1|\Lambda_0^2|} > 0.$$

Hence, we find

$$\left\{P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathscr{V}_2(t)+\rho_0\mathrm{E}(t)\right\}' \leq -\lambda_7\vartheta(t)P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\frac{\mathrm{E}(t)}{\mathrm{E}(0)}$$
$$= -\lambda_7\vartheta(t)\mathcal{H}\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right) \quad \forall t\geq t_0. \tag{65}$$

Now, we take the following

$$\mathscr{V}_3(t)=\begin{cases}\mathscr{V}_1(t), & \text{if } P \text{ is linear on } \quad [0,\omega],\\ P'\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)\mathscr{V}_2(t)+\rho_0\mathrm{E}(t), & \text{if } P'(0)=0 \quad \text{and} \quad P''>0 \quad \text{on } (0,\omega],\end{cases}$$

Then, from (59) and (65), we get

$$\mathscr{V}_3'(t)\leq -\lambda_8\vartheta(t)\mathcal{H}\left(\omega_0\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right) \quad \forall t\geq t_0. \tag{66}$$

Since $\mathscr{V}_3(t)\sim \mathrm{E}(t), \exists\aleph_1,\aleph_2>0$ such that

$$\aleph_1\mathscr{V}_3(t)\leq \mathrm{E}(t)\leq \aleph_2\mathscr{V}_3(t). \tag{67}$$

Introducing the functional

$$\mathscr{W}(t):=\aleph_1\frac{\mathscr{V}_3(t)}{\mathrm{E}(0)}, \tag{68}$$

by (67), we obtain

$$\mathscr{W}(t)\leq \frac{\mathrm{E}(t)}{\mathrm{E}(0)}<1. \tag{69}$$

From (66), (68) and (69) with the increasing nature of P, we have

$$\mathscr{W}'(t)\leq -\frac{\aleph_1\lambda_8}{\mathrm{E}(0)}\vartheta(t)\mathcal{H}\left(\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right)$$
$$\leq -\lambda_9\vartheta(t)\mathcal{H}\left(\frac{\mathrm{E}(t)}{\mathrm{E}(0)}\right). \tag{70}$$

Integrating (70) over (t_0,t) and utilizing $P'(t)=-(1/\mathcal{H}(t))$, we obtain

$$P(\mathscr{W}(t))-P(\mathscr{W}(0))\geq \lambda_9\int_{t_0}^{t}\vartheta(\varsigma)\,\mathrm{d}\varsigma. \tag{71}$$

The decreasing nature of P^{-1} implies

$$\mathscr{W}(t)\leq P^{-1}\left(P(\mathscr{W}(0))+\lambda_9\int_{t_0}^{t}\vartheta(\varsigma)\,\mathrm{d}\varsigma\right), \quad \forall t\geq t_0. \tag{72}$$

As a result, the equivalence $\mathscr{W}(t)\sim \mathrm{E}(t)$, gives (49) which completes the proof.

■

4. Conclusion

The primary motivation for this study was to investigate the asymptotic behaviour of solutions to viscoelastic wave equations with nonlinear distributed delay in the boundary feedback, particularly under acoustic boundary conditions. This framework represents a significant class of mathematical models commonly employed in applied and experimental sciences, especially in viscoelasticity theory. Future research aims to extend this work by incorporating additional damping mechanisms and terms, such as Balakrishnan-Taylor damping, dispersion effects, and logarithmic corrections, with a focus on nonlinear settings.

Authors' contributions

All authors contributed equally and significantly in writing this paper and typed, read, and approved the final manuscript.

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Availability of data and materials

All data regarding the research work is mentioned in the research work.

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