MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University, M'sila

Faculty of Technology

Department of Mechanical Engineering

Course Handout

Fluid Mechanics 2

Intended for students Third year Bachelor's degree Energetics

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Preface

This handout complies with the ministerial outline for the Fluid Mechanics 2 course offered in third year S5_ Bachelor's degree LMD at Algerian universities. It is a continuation of the course given second year S3_ Bachelor's degree LMD .

Fluid mechanics is one of the most difficult disciplines to assimilate. It requires both theoretical mathematical knowledge (tensor calculus, divergence and gradient operators......) and basic physics (Archimede force, Bernoulli equation, notion of similarity, etc.).

This course is structured in three chapters. The first deals with fluid kinematics, in which theoretical mathematical knowledge is first used (differential equations, divergence and gradient operators, integrals), the conservation of mass equation and 2D plane flows in the incompressible and irrotational case, as well as the complex potential function.

The second chapter deals with integral conservation laws. The Reynolds transport theorem is applied to the equation of continuity of momentum and energy.

The notion of dimensional analysis and similarity seems essential and this is the subject of the third chapter.

TABLE OF CONTENTS

Chapter I
Fluid kinematics

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I.1 Introduction

The study of fluid mechanics includes:

- **Fluid statics**, in which we study the fluid at rest (course S3) and the essential law is the fundamental relation of statics.

- **Fluid kinematics**, is the analytical description of a system in motion. In other words, we're interested in the movements of fluids in relation to time, independently of the causes that provoke them, i.e. without taking into account the forces that are at their source.

- **Fluid dynamics**, in which fluid motion is studied in the context of interacting forces.

I.2 Mathematical concepts for fluid mechanics

I.2.1 Differential of a function

Given a function f which depends on the variables x, y and z, f=f(x ,y, z)

The total differential df is written:

$$\mathbf{df} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} et \frac{\partial f}{\partial z}$ Are the partial derivatives of f with respect to x, y and z

I.2.2 Vector analysis operators

- **Operator Nabla**

$$\vec{\nabla} = \begin{cases} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{cases}$$

- **Gradient of a scalar field**

$$\overrightarrow{\text{grad}}(f) = \vec{\nabla} f = \begin{cases} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{cases}$$

- **Divergence of a vector field**

$$\text{div}(\vec{V}) = \vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

- **Rotation of a vector field**

$$\overrightarrow{\text{rot}}(\vec{V}) = \vec{\nabla} \wedge \vec{V} = \left| \begin{array}{l} \dfrac{\partial V_z}{\partial y} - \dfrac{\partial V_y}{\partial z} \\ \dfrac{\partial V_x}{\partial z} - \dfrac{\partial V_z}{\partial x} \\ \dfrac{\partial V_y}{\partial x} - \dfrac{\partial V_x}{\partial y} \end{array} \right.$$

- **Laplacian of a function**

$$\Delta\phi = \nabla^2\phi = \text{div}(\overrightarrow{\text{grad}}\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

I.3 Description of a moving fluid

I.3.1 The fluid particle

The fluid particle is chosen as the elementary entity for a complete description of flows: It is a "packet" of molecules surrounding a given point M; these are then assumed to all have the same velocity at the same instant.

In the study of fluid motion, we generally define at each point M: velocity, density ρ and pressure P (and possibly temperature T). Describing the motion of a fluid calls on notions that differ from those developed in point or solid mechanics. Fluid motion is a flow in which there is continuous deformation of the fluid. In a similar way to solid mechanics, we can isolate (by thought or by finding a means of visualization, coloring for example) a restricted part of the fluid called a particle and "follow" it over time, i.e. know its position at each instant. This position will be known, for example, by its Cartesian coordinates x (t,x_0, y_0, z_0), $y(t,x_0, y_0, z_0)$ and $z(t,x_0, y_0, z_0)$ where x_0,$y_0$and z_0 represent the coordinates of the selected particle at time t_0.

The particle's velocity will have the following components:

$$u = \frac{\partial x}{\partial t}, v = \frac{\partial y}{\partial t} \text{ et } \quad w = \frac{\partial z}{\partial t} \qquad \text{(I.1)}$$

The velocity of the fluid particle is then defined by :

$$\vec{V} = \begin{pmatrix} u = \dfrac{\partial x}{\partial t} \\ v = \dfrac{\partial y}{\partial t} \\ w = \dfrac{\partial z}{\partial t} \end{pmatrix} = \vec{V}(r_0, t) \qquad \text{(I.2)}$$

Different types of fluid flow regimes can be observed.

- **Permanent (or stationary) regime**: quantities do not depend on time $\frac{\partial}{\partial t} = 0$ $\vec{V} = \vec{V}$ (M) (so for ρ and P)(this does not mean that the fluid has a constant velocity everywhere, only that the fluid velocity at a given point is the same at every instant.
- **Uniform regime**: velocity does not depend on the point considered = (t)
- **Laminar regime**: fluid layers slide relative to each other, velocities are continuous.
- **Turbulent regime**: velocities are discontinuous, fluid layers interpenetrate aleatory.

I.3.2 Lagrange description - Euler description

The fluid in motion can be described in two equivalent ways. We can choose to follow the fluid particles as they move (Lagrange point of view) and the variables $r_0=(x_0, y_0, z_0)$ and t are called Lagrange variables.

The Lagrange point of view consists in focusing on the trajectory of the fluid particles.

We can take a snapshot at a given instant of the velocity field of all fluid particles (Euler's point of view). Euler's point of view focuses on the evolution of fluid properties at different points and over time.

Lagrange's method proves tricky in most cases, since it's not easy to keep track of the particles: it's rarely used.

Euler's method consists in knowing the velocity of the particles over time t at a given point determined by its coordinates, for example Cartesian x, y and z. The three projections of the velocity of the fluid particle passing through point M at time t are called Euler variables. This method is more widely used than Lagrange's, as knowledge of the velocity field is sufficient to describe the fluid in motion.

I.4 Trajectories and path lines

I.4.1 Trajectory:

The trajectory of a fluid particle is defined by the path followed by this particle in the course of time, i.e. the set of successive positions of this particle in the motion.

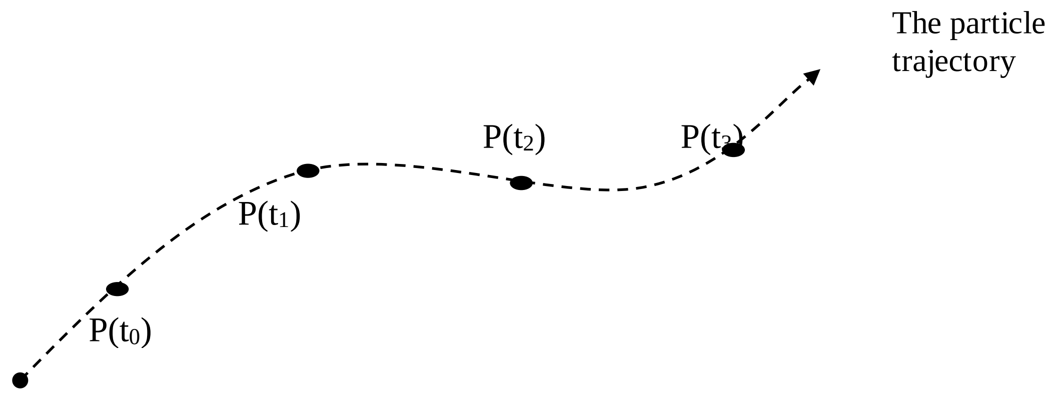


Figure 1 Particle trajectory

The trajectory can be visualized by injecting a drop of dye and following its movement. Trajectories are generally calculated by eliminating time from the expressions expressing the position of a fluid particle at each instant:

$$\overrightarrow{OM} = \vec{r}(t) = (x(t), y(t), z(t))$$

If we know the velocity in Eulerian description, we can determine the particle trajectories by integrating this velocity with respect to time.

Given the velocity:

$$\vec{V}(r,t) = \vec{V}(x,y,z,t) = \begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{pmatrix} \text{ in Eulerian description}$$

By definition

$$\vec{V} = \frac{d\vec{r}}{dt} = \begin{pmatrix} \dot{x} = \dfrac{dx}{dt} \\ \dot{y} = \dfrac{dy}{dt} \\ \dot{z} = \dfrac{dz}{dt} \end{pmatrix} \tag{I.3}$$

This gives us the differential system: $$\begin{pmatrix} \dfrac{dx}{dt} = u(x,y,z,t) \\ \dfrac{dy}{dt} = v(x,y,z,t) \\ \dfrac{dz}{dt} = w(x,y,z,t) \end{pmatrix} \tag{I.4}$$

Integrating this system with initial conditions r_0 =(x_0=x(t_0), y_0= y(t_0), z0=z(t_0)), we obtain the position at each instant $\vec{r}(t) = (x(t), y(t), z(t)) = \vec{r}_0 + \int_0^t \vec{V}(r_0, t)dt$

By eliminating time, we obtain a relationship between the variables (x, y, z) corresponding to the equation of the particle's trajectory.

I.4.2 Streamlines:

Let's adopt Euler's approach and assume that at each instant t we know the velocity vector of a fluid particle located at M. The velocity vector then designates a vector field.

$\vec{V}(M,t)$.

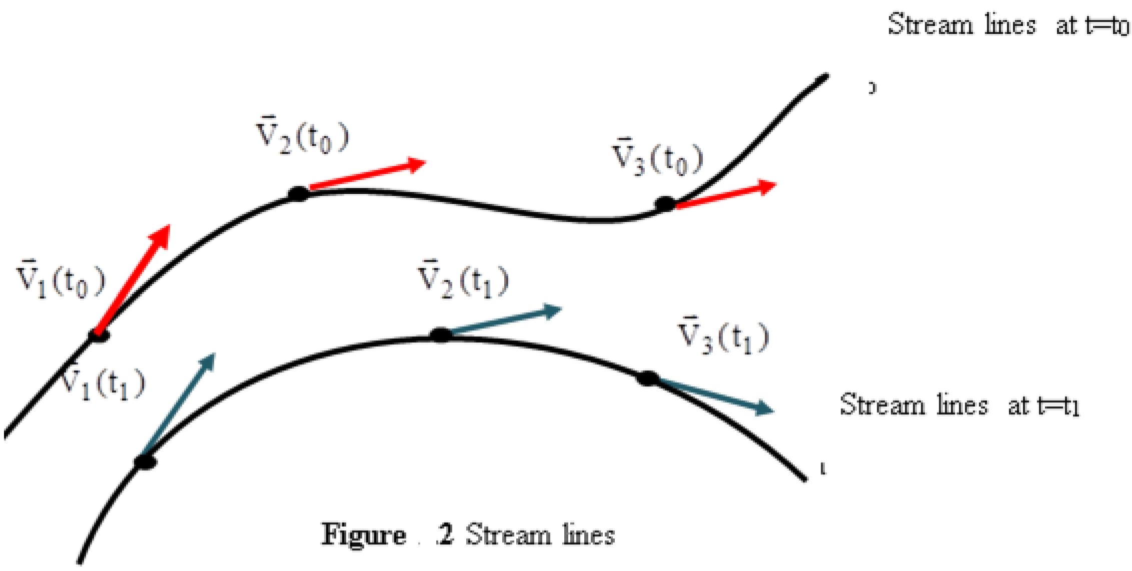


Figure . 2 Stream lines

By definition, a **streamline**, or flow line, is a field line of the velocity vector, i.e. a curve C such that at a fixed instant t, for any point M ∈C, $\vec{V}(M,t)$ is tangent to C at M. When the velocity field does not depend on time, flow lines do not evolve over time: the flow regime is said to be **stationary** or **permanent.**

Let $\overrightarrow{dM}$ a flow line element, $\overrightarrow{dM} = (dx, dy, dz)$, $\overrightarrow{dM}$ is parallel at M to the velocity:

At the velocity $\vec{V}(M,t)$, $\overrightarrow{dM} // \vec{V} \Leftrightarrow \overrightarrow{dM} \wedge \vec{V} = 0$

Or

$$\vec{V}(M,t) = \vec{V}(x,y,z,t) = \begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{pmatrix} \Rightarrow d\overrightarrow{M} \wedge \vec{V} = \begin{pmatrix} wdy - v\ dz \\ udz - wdx \\ vdx - udy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{(I.5)}$$

Finally, we obtain the relationships defining the stream lines

$$\frac{dx}{u(x,y,z,t)} = \frac{dy}{v(x,y,z,t)} = \frac{dz}{w(x,y,z,t)} \quad \text{(I.6)}$$

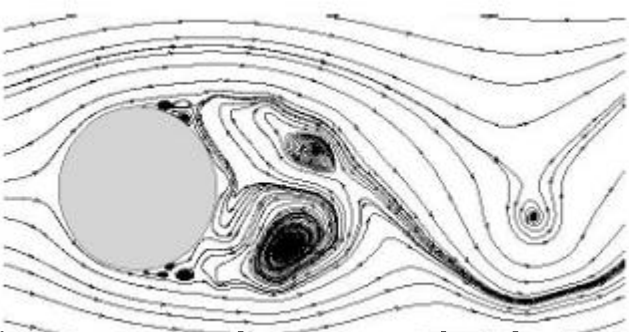


Figure I.3 Stream lines over an obstacle

Note

- Streamlines are generally time-dependent, so they deform over time.
- In steady state (stationary flow), velocities no longer depend on time, and the two previous conditions coincide with :

$$\frac{dx}{u(x,y,z)}=\frac{dy}{v(x,y,z)}=\frac{dz}{w(x,y,z)} \tag{I.7}$$

The particles continuously follow the same trajectories, generating the same streamlines. In this particular case, trajectory and streamlines are identical.

Other quantities characterizing fluid motion can also be defined:

I.4.3 Current tube:

A **current tube** is defined as the set of current lines supported by a closed contour.

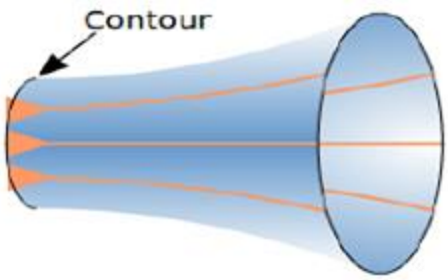


Figure 4 Current tube

I.4.4 Emission lines:

Emission lines are the set of all particles having coincided at an earlier instant with a fixed point E.

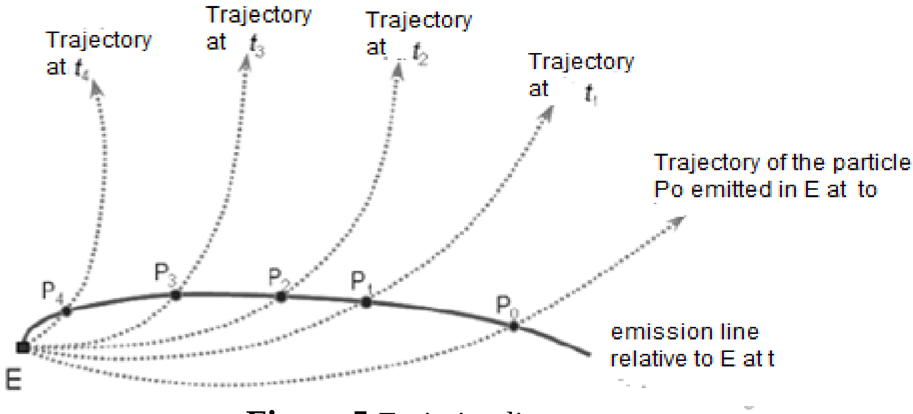


Figure 5 Emission lines

To visualize emission lines, we can inject dye continuously at point E. The colored curves correspond to the emission lines.

I.5 Particular derivative

Consider a local physical quantity G(M, t) attached to a fluid particle located in M at time t . We can think of temperature, pressure, density…. Let's calculate the rate of change of this quantity as we follow the particle. This quantity is called the particular derivative and is denoted DG/Dt.

The fluid particle at time t+dt will be at the point with coordinates x+udt, y+vdt, z+wdt

The variation of the function G will therefore be equal to:

$$dG = G(x+udt, y+vdt, y+wdt) - G(x,y,z) = \frac{\partial G}{\partial x}udt + \frac{\partial G}{\partial y}vdt + \frac{\partial G}{\partial z}wdt + \frac{\partial G}{\partial t}dt$$

The derivative $\frac{\mathbf{dG}}{\mathbf{dt}}$, denoted $\frac{\mathbf{DG}}{\mathbf{Dt}}$ and called the particular derivative, is equal to :

$$\frac{DG}{Dt} = \frac{dG}{dt} = \frac{\partial G}{\partial x}u + \frac{\partial G}{\partial y}v + \frac{\partial G}{\partial z}w + \frac{\partial G}{\partial t}dt = \vec{V}\,\overrightarrow{grad}G + \frac{\partial G}{\partial t} \qquad \text{(I.8)}$$

This derivative appears as the sum of two terms:

- The first, called convective or advective, is due to the non-uniformity of the flow,
- The second, called temporal, is due to the unsteady nature of the flow.

I.6 Particle acceleration

Let's calculate the acceleration of a fluid particle from the Eulerian velocity field $\vec{V}(M,t)$. Acceleration is the rate of change of the velocity field as it follows a fluid particle. We There fore we have:

Calculons l'accélération d'une particule de fluide à partir du champ de vitesse Eulérien $\vec{V}(M,t)$. L'accélération est le taux de variation du champ de vitesse en suivant une particule de fluide. On a donc :

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{Du}{Dt}u + \frac{Dv}{Dt}v + \frac{Dw}{Dt}w \tag{I.9}$$

The velocity $\vec{V} = \begin{pmatrix} u = \frac{dx}{dt} \\ v = \frac{dy}{dt} \\ w = \frac{dz}{dt} \end{pmatrix}$,

$$\vec{a} = \begin{pmatrix} a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial t} \\ a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial t} \\ a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial t} \end{pmatrix} \tag{I.10}$$

$$\vec{a} = \begin{pmatrix} a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix} \tag{I.11}$$

This gives :

$$\begin{pmatrix} a_x = \dfrac{Du}{Dt} = \dfrac{\partial u}{\partial t} + (\vec{V}.\vec{\nabla})u \\ a_y = \dfrac{Dv}{Dt} = \dfrac{\partial v}{\partial t} + (\vec{V}.\vec{\nabla})v \\ a_z = \dfrac{Dw}{Dt} = \dfrac{\partial w}{\partial t} + (\vec{V}.\vec{\nabla})w \end{pmatrix} \quad (I.12)$$

The acceleration can be broken down as follows:

- The first term ($\partial/\partial t$): is related to the non-permanent nature of the velocity. It is called the local term.
- The second term $\vec{V}.\vec{\nabla}$: the convective derivative indicates the non-uniform nature of the velocity. It is called the convective term.

I.7 Volume flow and mass flow

To solve problems in fluid mechanics and hydraulics, we often use the concepts of flow rate and mean flow velocity.

Volume flow q_v measured in (m^3/s) or (l/s)

Mass flow rate q_m measured in (kg/s)

Volume flow rate is the volume of fluid δv passing through a given area per unit time (m^3/s).

$$\delta v_{trav} = q_v dt \quad (I.13)$$

The total volume traversing the surface considered during a period of time (t_2-t_1) is given by:

$$v_{trav} = \int_{t_1}^{t_2} q_v dt \quad (I.14)$$

The discharge for a constant vertical velocity on a section of pipe or duct (perfect fluid) is as follows:

$$q_v = V.S \quad (I.15)$$

$$\mathbf{q_m = \rho V.S = \rho.q_v} \quad (I.16)$$

(ρ the density of the fluid)

- **Expression of q_v as a function of the velocity field on the surface**

The volume flow rate is the flow of the vector across the surface considered:

$$q_v = \oiint_S \vec{V}.\vec{n}ds \qquad (I.17)$$

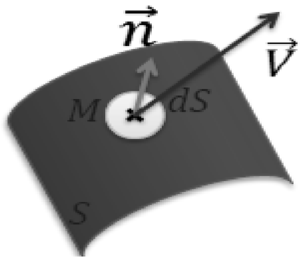


Figure 6 Velocity flux through a surface

- If the flow is in the same direction as the surface normal vector: **qv**> 0, otherwise **qv**<0
- The mass flow rate is the mass of fluid passing through a given surface per unit of time ($Kg.S^{-1}$).

$$\delta m_{trav} = q_m dt \qquad (I.18)$$

- The total mass passing through the surface in question over a period of time (t_2-t_1) is given by :

$$m_{trav} = \int_{t_1}^{t_2} q_m dt \qquad (I.19)$$

The mass flow rate is **the vector flux** passing through a given surface:

$$q_m = \oiint_S \rho\vec{V}.\vec{n}ds \qquad (I.20)$$

- If the flow is in the same direction as the surface normal vector **q_m>0**, otherwise **q_m<0**

The $\rho\vec{V}$ field thus appears as the mass current density, or surface mass flow.

- In the particular case of a permanent conservative flow through a current tube, the mass flow rate is conserved: **$q_{m1} = q_{m2}$**
- If the fluid is also incompressible: **$q_{v1} = q_{v2}$**

I.8 Continuity equation

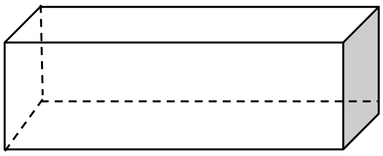
The continuity equation translates the principle of conservation of mass:

The change in mass over time dt of a fluid volume element dv = dx dy dz must be equal to the sum of the masses of incoming fluid, minus that of outgoing fluid.

On considère alors un élément de volume de fluide *dv*

dv=dx.dy.dz

The mass $\boldsymbol{m}=\iiint_v \boldsymbol{\rho dv}$ of a portion of fluid volume bounded by a surface (S) that we follow in its motion remains constant, so its particle derivative is zero.



(I.21)

$$\frac{dm}{dt}=\frac{d}{dt}\iiint_v \rho dv=\iiint_v \frac{\partial \rho}{\partial t}dv+\iint_S \rho(\vec{V}.\vec{n})dS=0 \qquad \text{(I.21)}$$

Local derivative

Convective derivative

I.8.1 Green-Ostrogradsky theorem or divergence theorem

The flux of a vector field $\vec{A}(M)$ through a closed surface (*S*) is equal to the integral over the volume (*v)* bounded by (S) of the divergence of the vector field.

$$\iint_S \vec{A}(M).\vec{n}.dS=\iiint_v div\vec{A}(M).dv$$
$$\text{et} \quad div\vec{A}=\vec{\nabla}.\vec{A} \qquad \text{(I.22)}$$

So we can write :

$$\iint_S \rho(\vec{V}.\vec{n}).dS=\iiint_v div(\rho\vec{V}).dv=\iiint_v (\vec{\nabla}.\rho\vec{V})dv \qquad \text{(I.23)}$$

$$\frac{dm}{dt}=\iiint_v \frac{\partial \rho}{\partial t}dv+\iiint_v div(\rho\vec{V})dv=0 \qquad \text{(I.24)}$$

Or still :

$$\iiint_v \left(\frac{\partial \rho}{\partial t}+\vec{\nabla}.\rho\vec{V}\right)dv=0 \qquad \text{(I.25)}$$

Then : On an arbitrary volume (the integral must be zero) this relationship becomes :

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\rho\vec{V} = \frac{\partial \rho}{\partial t} + \mathrm{div}\,\rho\vec{V} = 0 \qquad (I.26)$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\rho\vec{V} = 0 \qquad (I.27)$$

Is the continuity equation

In Cartesian coordinates, this equation is written :

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

This is the general continuity equation, applicable to all types of flow, and all types of compressible and incompressible fluids.

If the fluid is in permanent motion, the density is independent of $\frac{\partial \rho}{\partial t} = 0$ time, and this becomes:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad or \quad div(\rho\vec{V}) = 0$$

The equation obtained indicates that the $\rho\vec{V}$ flow through the closed surface is zero (**conservation of mass flow**).

$$\iint_S = \iiint_v \mathrm{div}(\rho\vec{V}).dv = \iint \rho(\vec{V}.\vec{n}).dS = 0$$

For a two-dimensional plane flow we write :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

For one-dimensional flow in the x direction

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = cte \Rightarrow q_v = uS = cte$$

(S flow cross-section)

Special case of an incompressible fluid :

In this case the density is **ρ =cte**

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0 \Rightarrow \mathrm{div}\vec{V} = 0$$

So the continuity equation reduces to : $\mathrm{div}\vec{V} = 0$ (I.28)

I.8.2 Divergence of a velocity field

I.8.2.1 Definition :

Velocity field divergence ($div\vec{V}$) is a differential operator with scalar values that measures changes in the volume of a continuous medium. A positive (resp. negative) value is associated with expansion (resp. compression). In Cartesian coordinates, it is written :

$$\mathrm{div}\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{\partial v_i}{\partial x_i}$$

In cylindrical coordinates, it is written :

$$\mathrm{div}\vec{V} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

We can say that the divergence of the velocity field gives us information about the change in volume of a fluid element we're following as it moves. If this element maintains a constant volume, the divergence is zero. If this is true at any point in the fluid, then the volume of all fluid elements will remain constant throughout the flow: such a flow is said to be **incompressible.**

I.9 Some flow examples

I.9.1 Uniform Flow

In the absence of deformation and rotation, the flow is said to be uniform. This movement corresponds to solid translational motion.

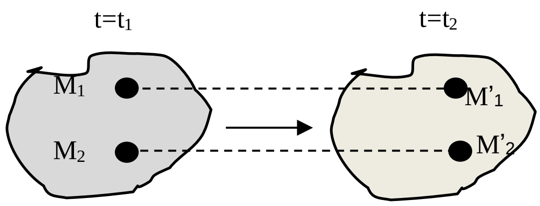


Figure 7 Uniform flow without deformation or rotation

The pure rotational movement takes place without deformation and is therefore comparable to solid rotation, as shown in the following figure.

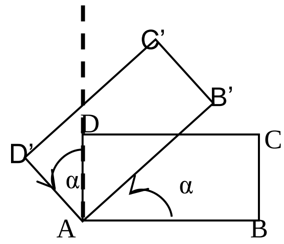


Figure 8 Fluid flow rotation without deformation

I.9.2 Rotational Flow

The rotational velocity field of a flow $\overrightarrow{rot}\vec{V}$ is a vector-valued differential operator that measures twice the rate of rotation of fluid particles on themselves.

In cartesian coordinates, the vortex vector is written as:

$$\vec{\boldsymbol{\Omega}} = \overrightarrow{rot}\vec{V} = \begin{pmatrix} \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{pmatrix} = \vec{\nabla}\Lambda\vec{V} \tag{I.29}$$

A rotational flow is characterized by the vortex vector Ω such that:

$$\vec{\boldsymbol{\Omega}} = 2\vec{\omega} = \vec{\nabla}\Lambda\vec{V} \tag{I.30}$$

And ω is the rate of rotation

In cylindrical coordinates with $\vec{V}(u_r, u_\theta, u_z)$,we have :

$$\vec{\Omega} = \left| \begin{array}{l} \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta} - \dfrac{\partial u_\theta}{\partial z} \\ \dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r} \\ \dfrac{1}{r}\left|\dfrac{\partial (r u_\theta)}{\partial r} - \dfrac{\partial u_r}{\partial \theta}\right. \end{array} \right. \tag{I.31}$$

For a plane flow, this vector has only one non-zero component since

ω =0 and u and v do not depend on z :

$$\overrightarrow{\mathrm{rot}}\vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{e}_z \qquad \text{(I.32)}$$

I.9.3 Stream Function–Incompressible flow

I.9.3.1Definition:

If the flow of an incompressible fluid is conservative, then the continuity equation is: $\vec{\nabla}.\vec{V} = 0$ (eq (I.28)).

If we put $\vec{V} = \vec{\nabla}\Lambda\vec{A}$ $,\forall$ $\vec{A}$ $then$ $\vec{\nabla}.(\vec{\nabla}\Lambda\vec{A}) = 0$

$\vec{A}$ Is called potential vector

In Cartesian coordinates:

$$\vec{V} = \vec{\nabla}\Lambda\vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{vmatrix} \Lambda \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = u \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = v \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = w \end{vmatrix} \qquad \text{I.33)}$$

If we consider a flow in the plane $\perp$ to *Oz*, and therefore invariant by translation along *z*, then: $w = 0 \;\; et \;\; \frac{\partial}{\partial z} = 0$ from which :

$u = \frac{\partial A_z}{\partial y} \, et \, v = -\frac{\partial A_z}{\partial x}$ then : $A_z(x,y) = \psi(x,y)$, the function ψ is called **stream function**. therefore :

$$\begin{cases} u = \frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \psi}{\partial x} \end{cases} \qquad \text{(I.34)}$$

Is the velocity field in Cartesian coordinates.

In cylindrical coordinates, this velocity field is written as:

$$\begin{cases} u_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ u_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases} \text{ Or } \boldsymbol{\psi(r, \theta)} \tag{I.35}$$

I.9.3.2 Properties of the stream function

As we posed $\vec{\nabla}.\vec{V} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ et $u = \dfrac{\partial\psi}{\partial y}, v = -\dfrac{\partial\psi}{\partial x}$

Then:

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial y\partial x}$$

This relationship constitutes **Schwartz's** theorem**.** And so dψ is an exact total differential:

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$$

In the plane (x, y), the set of points for which the value of ψ is constant **ψ(x,y) = cte** corresponds to the curve **y(x)** along which **dψ=0**

On this curve, check that:

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -vdx + udy = 0 \tag{I.36}$$

Or : $-vdx + udy = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{v}{u}$

ψ(x,y) = cte then **y(x)** is as :

$$\underbrace{\frac{dy}{dx}}_{\text{pente dela courbe } y=f(x)} = \underbrace{\frac{v}{u}}_{\text{pente du vecteur vitesse } \vec{V}} \tag{I.37}$$

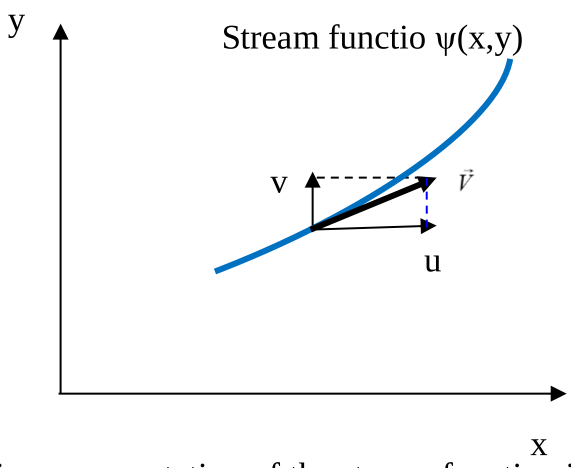


Figure 9 Qualitative representation of the stream function in the (x, y)plan

Let's calculate the flow between two infinitely adjacent current lines:

Let ψ(x,y) be the current function L and ψ+dψ the adjacent current function M. The velocity vector V is perpendicular to the line AB and has components u and v in the x and y directions.

We know that the flow rate $q_v = \int \vec{V}.\vec{n}ds$

Flow through AB= flow through AO+flow through OB

Vds= udy-vdx

$$Vds = \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx = d\psi$$

And so **dq_v** =dψ therefore, between any two current lines of constants ψ_A and ψ_B :

$$q_v = \int_A^B dq_v = \int_A^B d\psi = \psi_B - \psi_A \qquad (I.38)$$

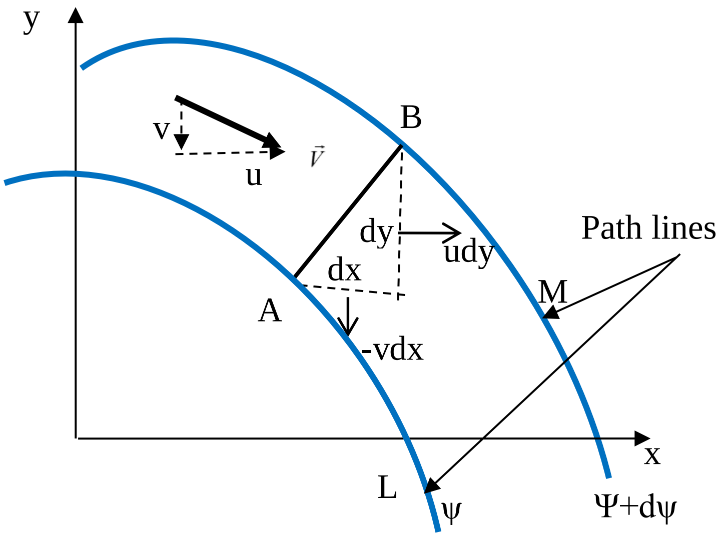


Figure 10 Flow two points and its relationship to the stream function

I.9.4 Irrotational flow – Velocity potential

I.9.4.1 Definition:

Flow is said to be **irrotational** when the fluid particles do not undergo **pure rotations**:

Ω=0, i.e. $\overrightarrow{rot}\vec{V} = 0$

$$\Omega = \begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix} = 0 \quad \Rightarrow \quad \left.\begin{matrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{matrix}\right\} = \vec{0}$$

$$\Rightarrow \quad \vec{\omega} = \tfrac{1}{2}\vec{\nabla} \wedge \vec{V} = \vec{0}$$

In other words, the rotation rate **ω** is zero in an irrotational flow.

Or, from the mathematical relation $\vec{\nabla} \wedge (\vec{\nabla}\varphi) = \vec{0},\ \forall\varphi$

We can define a scalr φ such that : $\vec{V} = \vec{\nabla}\varphi$, **φ** is called the velocity potential.

In the cartesian reference frame and considering plan flow, we can write:

$$\vec{V} = \vec{\nabla}\varphi \quad \Rightarrow \quad u = \frac{\partial\varphi}{\partial x} \quad , \quad v = \frac{\partial\varphi}{\partial y} \quad \textbf{et} \quad w = \frac{\partial\varphi}{\partial z} \qquad \text{(I.39)}$$

If we assume that the fluid is incompressible, we must verify :

$$\vec{\nabla}.\vec{V} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This leads to the relationship :

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0 \qquad \Rightarrow \quad \Delta\varphi=0 \qquad \textit{\textbf{Laplace equation}}$$

We therefore conclude that the velocity potential must satisfy ***Laplace*'s equation**.

Note :
If the flow is irrotational, the stream function must also satisfy ***Laplace*'s equation :**

$$\vec{V}=\begin{cases}\partial\Psi/\partial y\\ -\partial\Psi/\partial x\\ 0\end{cases} \quad \text{et } \underbrace{\vec{\nabla}\wedge\vec{V}=\vec{0} \quad \Rightarrow \begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ 0\end{pmatrix}\wedge\begin{pmatrix}\partial\Psi/\partial y\\ -\partial\Psi/\partial x\\ 0\end{pmatrix}=\vec{0}}$$

$$\Delta\Psi=0 \quad \Leftarrow \quad -\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi}{\partial y^2}=0$$

I.9.4.2 Properties of the velocity potential

When the flow is plan the equation $\varphi(x,y)=C^{te}$ defines in the plan flow a curve called « **equipotential** ».

A long of this curve, since $\varphi(x,y)=C^{te}$, we must verify : $d\varphi=0$

Or, la différentielle peut s'écrire : $\mathbf{d}\varphi=\frac{\partial\varphi}{\partial x}\mathbf{d}x+\frac{\partial\varphi}{\partial y}\mathbf{d}y$

And as along an equipotential $\mathbf{d}\varphi=\mathbf{0}$, then :

$$d\varphi=\frac{\partial\varphi}{\partial x}\mathbf{d}x+\frac{\partial\varphi}{\partial y}\mathbf{d}y=0 \Rightarrow \quad u\mathbf{d}x+v\mathbf{d}y=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{u}{v} \qquad \text{(I.40)}$$

So $\frac{dy}{dx}=-\frac{u}{v}$ relationship to be verified at any point on the equipotential.

At any point M(x,y) in the flow plane, the ***streamline and equipotential are orthogonal***.

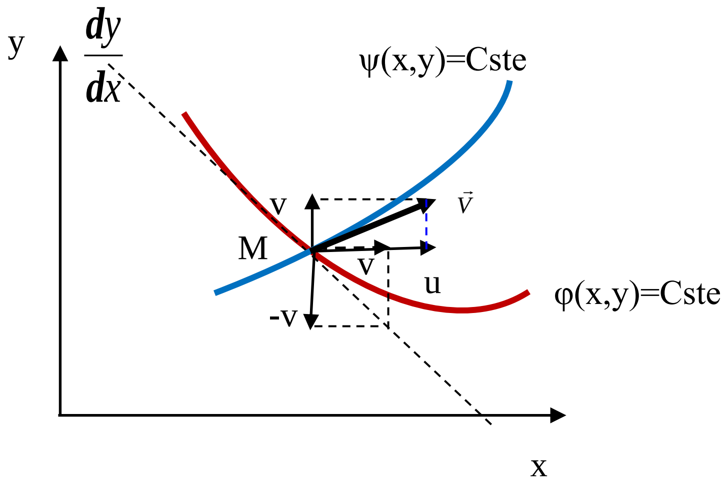


Figure 11 Qualitative representation of the path line and the equipotential in the (x, y) plan

I.9.4.3 Equations de Cauchy Riemann

We can conclude from what we have seen above that :

- The velocity potential(φ) exists only for an irrotational flow.
- The stream function(ψ) is applied for rotational and irrotational flow (stationary and incompressible).
- In the case of irrotational flow, the stream function and velocity potential both satisfy **Laplace's equation.**
- Therefore, for an irrotational and incompressible flow, the following relationship can be verified:

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned} \qquad (I.41)$$

- These equations are called **Cauchy-Riemann equations.**

I.9.4.4 Calculating the length of an arc element along a stream line

We want to calculate the arc on the stream line (ψ(x,y)=cste).

We have : $\mathbf{d}s_{\psi=C^{te}} = \sqrt{\mathbf{d}x^2 + \mathbf{d}y^2}$

Or :

$$\begin{aligned} \mathbf{d}\varphi &= \frac{\partial \varphi}{\partial x}\mathbf{d}x + \frac{\partial \varphi}{\partial y}\mathbf{d}y \\ &= u\mathbf{d}x + v\mathbf{d}y \end{aligned}$$

In addition, along the stream line we have **ψ(x,y)=cste ,** i.e.: $\frac{\mathbf{d}y}{\mathbf{d}x}=\frac{v}{u}$ therefore: $\boldsymbol{dy}=\frac{v}{u}\boldsymbol{dx}$

by replacing we then obtain

$$\mathbf{d}\varphi = u\mathbf{d}x + \frac{v^2}{u}\mathbf{d}x \;\; = \frac{u^2+v^2}{v}\mathbf{d}y \tag{I.42}$$

There fore : $\mathbf{d}y = \frac{v}{u^2+v^2}\mathbf{d}\varphi$

$$\mathbf{d}x = \frac{u}{u^2+v^2}\mathbf{d}\varphi$$

Then:

$$\mathbf{d}s_{\Psi=C^{te}} = \sqrt{\mathbf{d}x^2+\mathbf{d}y^2} = \sqrt{\frac{u^2+v^2}{\left(u^2+v^2\right)^2}\mathbf{d}\varphi^2} = \frac{\mathbf{d}\varphi}{\sqrt{u^2+v^2}} \tag{I.43}$$

So: $\mathbf{d}s_{\Psi=C^{te}} = \frac{\mathbf{d}\varphi}{V}$

- The distance between two equipotential is inversely proportional to the flow velocity.
- One of the properties of the current function is that the difference in the stream function between two points represents the fluid flow through any line joining the points.
- If two points lie in the same streamline, in this case there is no flow between these two points and therefore ψ -ψ_{12} =0 we then have **ψ(x,y)=cste**
- Similarly, **φ=cste** , represents the case where the velocity potential is the same at each point, and is said to represent an **equipotential line**.

Given two curves **φ=cste** and **ψ=cste ,** these two curves intersect at every point.

At the point of intersection of these curves, the slopes are:

For the curve **φ=cste : the slope** = $\frac{\partial y}{\partial x} = \frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial x}} = \frac{u}{v}$

For the curve **ψ=cste : the slope** = $\frac{\partial y}{\partial x} = \frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial x}} = \frac{-v}{u} = -\frac{v}{u}$

The product of the slopes of these curves is:

$$\frac{u}{v}\times-\frac{v}{u}=-1$$

This shows that equipotential lines and current lines form an orthogonal network at all points of intersection.

I.10 Flow representation by complex functions

Many classical plane flows can be represented by complex functions. Let $f(z)=\varphi(x,y)+i\psi(x,y)$ where z=x+iy is the complex variable associated with the complex potential function f(z) (φ and ψ represent the potential and stream functions respectively). For this function f(z) to be analytic, its derivative must be defined everywhere, i.e.

$\lim_{\Delta z\to 0}\left(\frac{\Delta f}{\Delta z}\right)$ tends towards the same value regardless of howΔ z tends towards 0.

If we put : $\Delta z\to 0 \quad \Leftrightarrow \quad \begin{cases}\Delta x\to 0\\ \Delta y=0\end{cases}$ **ou** $\begin{cases}\Delta x=0\\ \Delta y\to 0\end{cases}$

And , Δ z can be made to tend towards 0 in the following two ways:

therefore :

$$\lim_{\Delta z\to 0}\left(\frac{\Delta f}{\Delta z}\right)=\lim_{\substack{\Delta x\to 0\\ \Delta y=0}}\left(\frac{\Delta\varphi+i\,\Delta\Psi}{\Delta x+i\,\Delta y}\right)=\lim_{\substack{\Delta x=0\\ \Delta y\to 0}}\left(\frac{\Delta\varphi+i\,\Delta\Psi}{\Delta x+i\,\Delta y}\right)=\frac{\mathbf{df}}{\mathbf{dz}}$$

$$\lim_{\Delta x\to 0}\left(\frac{\Delta\varphi}{\Delta x}+i\frac{\Delta\Psi}{\Delta x}\right)\quad,\quad\lim_{\Delta y\to 0}\left(-i\frac{\Delta\varphi}{\Delta y}+\frac{\Delta\Psi}{\Delta y}\right)$$

$$\Downarrow\qquad\qquad\Downarrow$$

$$\frac{\partial\varphi}{\partial x}+i\frac{\partial\Psi}{\partial x}=\frac{\partial f}{\partial x}\qquad\qquad -i\frac{\partial\varphi}{\partial y}+\frac{\partial\Psi}{\partial y}=-i\frac{\partial f}{\partial y}$$

Then :

$$\frac{\partial\varphi}{\partial x}+i\frac{\partial\Psi}{\partial x}=-i\frac{\partial\varphi}{\partial y}+\frac{\partial\Psi}{\partial y}\ \text{, d'où : }\frac{\partial\varphi}{\partial x}=\frac{\partial\Psi}{\partial y}\quad\textbf{et}\quad\frac{\partial\varphi}{\partial y}=-\frac{\partial\Psi}{\partial x}$$

This system of equations constitutes the ***Cauchy-Riemann relations*** which verify the relations found above.

Finally, for $f(z)=\varphi(x,y)+i\,\Psi(x,y)$, to be an analytic function**.** $\varphi(x,y)$ et $\Psi(x,y)$ must verify these ***Cauchy.***

For a plan flow, which can be described by means of a stream function $\Psi(x,y)$ and a velocity potential $\varphi(x,y)$, these *Cauchy* relations are well verified:

$$u=\frac{\partial\varphi}{\partial x}=\frac{\partial\Psi}{\partial y} \quad \textbf{et} \quad v=\frac{\partial\varphi}{\partial y}=-\frac{\partial\Psi}{\partial x}$$

Consequently, the flow can also be described by means of the complex analytical function:

:

$$f(z)=\varphi(x,y)+i\,\Psi(x,y) \qquad \text{where} \qquad z=x+i\,y$$

This function is known *as the "**complex velocity potential**"*.

Properties :

We have seen that for a flow to be described by means of a stream function ψ and a velocity potential φ, these two functions must verify Laplace's equation ($\Delta\psi$=0 and $\Delta\varphi$=0).

Let there be two flows such that :

$$\begin{cases}\Delta\psi_1=0 \quad and \quad \Delta\varphi_1=0\\ \Delta\psi_2=0 \quad and \quad \Delta\varphi_2=0\end{cases} \Rightarrow \begin{cases} f_1(z)=\varphi_{1+i}\psi_1\\ f_2(z)=\varphi_{2+i}\psi_2\end{cases}$$

Since the Laplacian operator is linear, this implies that:

$$\begin{cases}\Delta(\lambda_1\psi_1+\lambda_2\psi_2)=\lambda_1\Delta\psi_1+\lambda_2\Delta\psi_2\\ \Delta(\lambda_1\varphi_{1+}\lambda_2\varphi_2)=\lambda_1\Delta\varphi_1+\lambda_2\Delta\varphi_2\end{cases}$$

We put : $\begin{cases}\psi=\lambda_1\psi_1+\lambda_2\psi_2\\ \varphi=\lambda_1\varphi_1+\lambda_2\varphi_2\end{cases} \Rightarrow \begin{cases}\Delta\psi=0\\ \Delta\varphi=0\end{cases}$

And then : $f(z)=\varphi_{+i}\psi \;=\lambda_1 f_1(z)+\lambda_2 f_2(z)$, f(z) describes the flow resulting from the superposition of the two flows f_1 and f_2 . Consequently, several elementary flows can be superimposed to create more complex flows, simply by adding the corresponding complex potentials.

I.10.1 Uniform flow

Consider the plane flow modeled by the complex velocity potential:

$f(z)=Uz$

We have : $\varphi(x,y)+i\,\Psi(x,y)=U(x+i\,y)=Ux+i\,Uy$

By identification, we obtain:

$$\varphi(x,y)=Ux$$
$$\Psi(x,y)=Uy$$

The stream lines are such that: $\Psi(x,y)=Uy=\mathbf{C}^{te}$

$\Rightarrow \quad y=\mathbf{C}^{te}\ \forall x$: these are horizontales lines

The equipotentials are such that: $\varphi(x,y)=Ux=\mathbf{C}^{te}$

$\Rightarrow \quad x=\mathbf{C}^{te}\ \forall y$ these are vertical lines

Determining the velocity field :

$$\vec{v} = \begin{cases} u = \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \Psi}{\partial y} = U \\ v = \dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \Psi}{\partial x} = 0 \end{cases}$$

The velocity is uniform : $\vec{v} = U\vec{e}_x$

Stream lines : $\Psi(x, y) = Uy = \mathbf{C^{te}} \Rightarrow y = \mathbf{C^{te}}\ \forall x$ (horizontal lines)

equipotentials : $\varphi(x, y) = Ux = \mathbf{C^{te}} \Rightarrow x = \mathbf{C^{te}}\ \forall y$ (vertical lines)

the velocity field :

$\vec{V} = U\vec{e}_x$

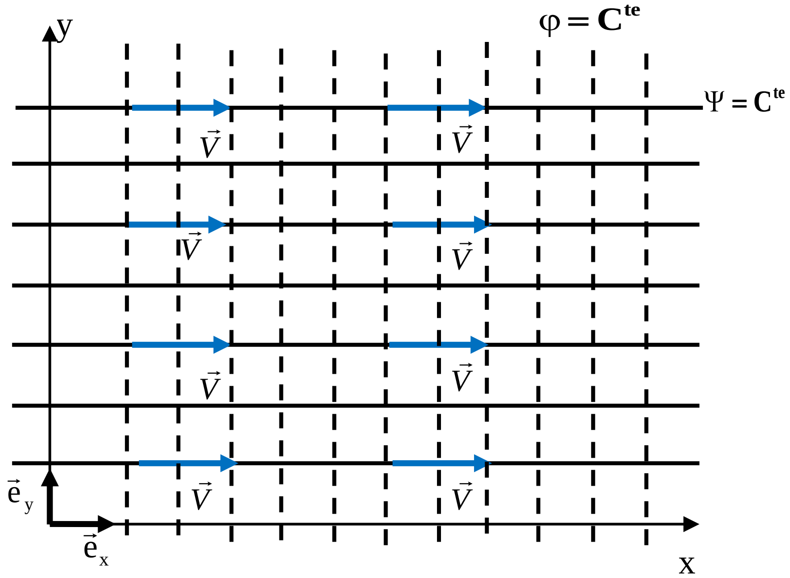


Figure 12 Uniform flow $f(z) = Uz$

I.10.2 Plane flow around a source or sink

Consider the plane flow modeled by the complex velocity potential:

$f(z) = \boldsymbol{C}\,\mathbf{ln}\,z$ where $z = x + i\,y = r\,e^{i\theta}$ et C is a real constant

$$\Rightarrow f(z) = \boldsymbol{C}\,\mathbf{ln}\left(r\,e^{i\theta}\right) = \boldsymbol{C}\left(\mathbf{ln}\,r + i\theta\right)$$

We can then deduce the current function and velocity potential :

$$\varphi(r,\theta) = \boldsymbol{C}\,\mathbf{ln}\,r$$
$$\Psi(r,\theta) = \boldsymbol{C}\,\theta$$

The stream lines are such as : $\Psi(r,\theta) = \boldsymbol{C}\,\theta = \boldsymbol{C}^{te}$

$\Rightarrow \theta = \boldsymbol{C}^{te}\ \forall r$ these are straight lines passing through the origin

The equipotential are such that: $\varphi(r,\theta) = C\ln r = C^{te}$

$\Rightarrow \quad r = C^{te} \ \forall\theta$ These are concentric circles centered on the origin.

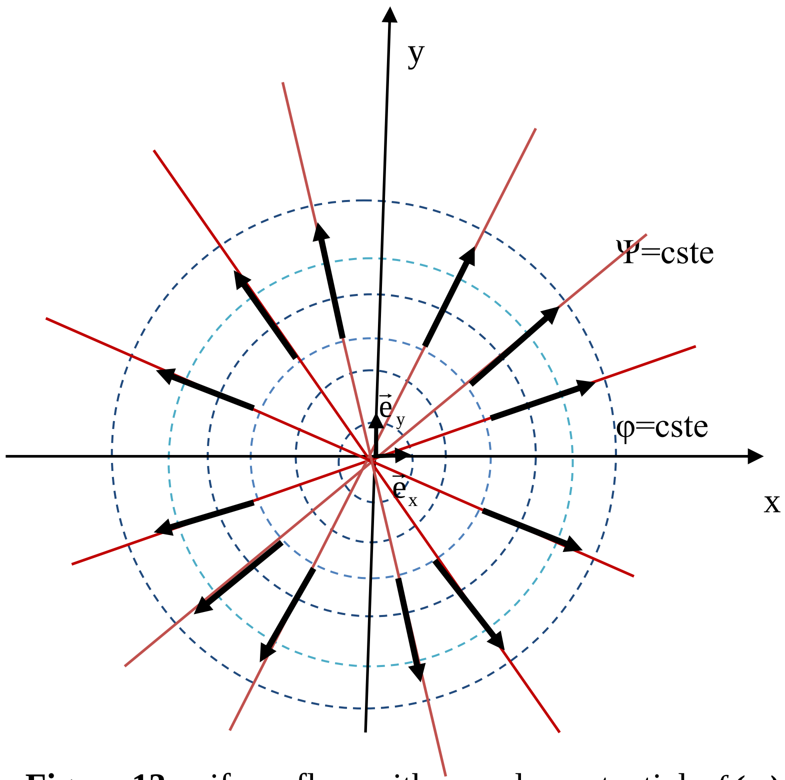


Figure 13 uniform flow with complex potential $f(z) = C\ln z$

Determining the velocity field :

$$\vec{v} = \begin{cases} v_r = \dfrac{\partial\varphi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\Psi}{\partial\theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta} = -\dfrac{\partial\Psi}{\partial r} \end{cases}$$

Or $\vec{v} = \begin{cases} v_r = C/r \\ v_\theta = 0 \end{cases} \Rightarrow \vec{V} = \dfrac{C}{r}\vec{e}_r$

The velocity is therefore radial and inversely proportional to distance from the origin.

If C>0, then flow is directed outwards

$\Rightarrow$ *Divergent* flow$\Rightarrow$ ***source*** at origin.

If C<0, then the flow is directed towards the origin

$\Rightarrow$ *convergent* flow$\Rightarrow$ ***sink*** at origin.

Physical meaning of the constant C :

The volume flow of this radial flow (source or sink) is calculated:

$$q_V = \oiint_S \vec{V}.\vec{n}\, dS$$ where S is a closed surface surrounding the origin.

$\vec{V} = \frac{C}{r}\vec{e}_r$ *andt* $\vec{n} = \vec{e}_r$

This is a linear flow taking place in the direction$\perp$ to the z axis, in the (xy) plane we can consider as the integration surface a cylinder of heightΔ z=1, and therefore :

$$\oiint_S \ldots dS = \oint_\ell \ldots \Delta z\, d\ell$$

Since the flow is on a plane, we integrate on a circle of any radius r centered on the origin.

$$q_V = \Delta z \oint_\ell \vec{V}.\vec{n}\, r\, d\theta = \Delta z\, r \int_0^{2\pi} \vec{V}.\vec{n}\, d\theta \text{ où } \begin{cases} \vec{V} = C/r\, \vec{e}_r \\ \vec{n} = \vec{e}_r \end{cases}$$

$$\Rightarrow q_V = \Delta z\, r \int_0^{2\pi} \frac{C}{r} d\theta = \Delta z\, r \frac{C}{r} \int_0^{2\pi} d\theta \qquad = 2\pi C \Delta z$$

1

volumetric flow rate per unit height

$$\Rightarrow C = \frac{q_V}{2\pi}$$ and therefore : $f(z) = \frac{q_V}{2\pi} \ln z$ **q_v>0** : source flow rate

q_v<0 : sink flow rate

I.10.3 Vortex or free vortex

Consider the plane flow modeled by the complex velocity potential:

$f(z) = -i\mathbf{C} \ln z$ where $z = x + iy = re^{i\theta}$ and C is a real constant

$$\Rightarrow f(z) = -iC \ln\left(re^{i\theta}\right) = -iC(\ln r + i\theta) = C\theta - iC \ln r$$

We can then deduce the current function and the velocity potential:

$$\begin{cases} \varphi(r,\theta) = C\,\theta \\ \Psi(r,\theta) = -C\,\ln r \end{cases}$$

The stream lines are such as : $\Psi(r,\theta) = -C\,\ln r = C^{te}$

$\Rightarrow \quad r = C^{te}\ \forall\theta$ these are concentric circles centred on the origin Les

the equipotentials are such that : $\varphi(r,\theta) = C\,\theta = C^{te}$

$\Rightarrow \quad \theta = C^{te}\ \forall r$ these are straight lines passing through the origin

Determining the velocity field :

$$\vec{V} = \begin{cases} v_r = \dfrac{\partial\varphi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\Psi}{\partial\theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta} = -\dfrac{\partial\Psi}{\partial r} \end{cases}$$

Soit : $\vec{V} = \begin{cases} v_r = 0 \\ v_\theta = \dfrac{C}{r} \end{cases} \Rightarrow \vec{V} = \dfrac{C}{r}\vec{e}_\theta$

Velocity is therefore ortho-radial and inversely proportional to distance from the origin.

If **C>0**, then the flow is around the origin in the ***trigonometric direction***.

If **C<0**, then the flow is ***clockwise*** around the origin.

Physical meaning of the constant C :

Let's calculate the velocity "circulation" around the origin:

$\Gamma = \oint_\ell \vec{V}.\mathbf{d}\vec{\ell}$ Where runs an arbitrary stream line, *i.e.* a circle of radius *r*.

With : $\vec{V} = \dfrac{C}{r}\vec{e}_\theta$ et $d\vec{\ell} = r\,d\theta\,\vec{e}_\theta \Rightarrow \quad \Gamma = \int_0^{2\pi} \dfrac{C}{r}\,r\,\mathbf{d}\theta = 2\pi C$

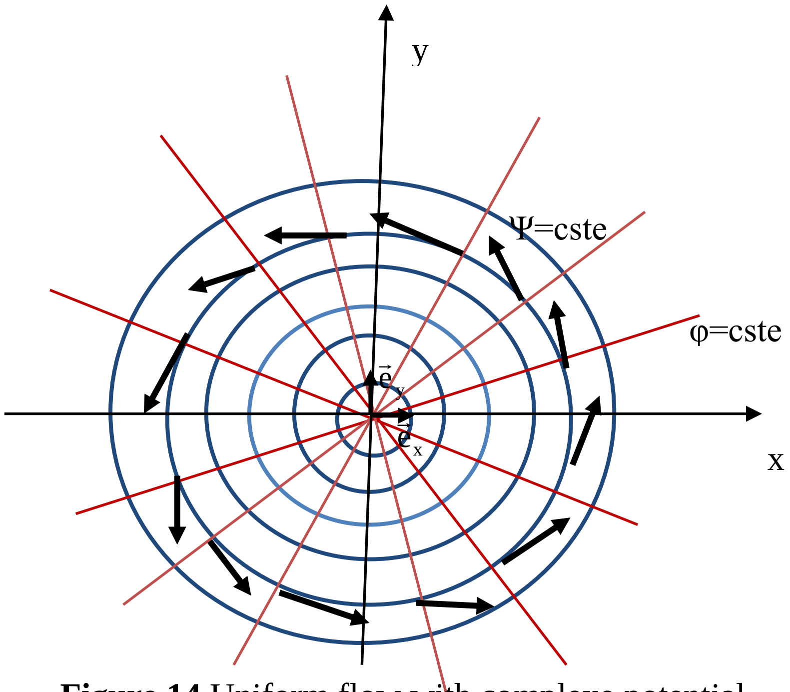


Figure 14 Uniform flow with complexe potential
$f(z) = -i\mathbf{C}\ln z$

and therefore $f(z) = -i\dfrac{\Gamma}{2\pi}\ln z$ whereΓ is the vortex **circulation** (free vortex).

so $\boldsymbol{C} = \dfrac{\boldsymbol{\Gamma}}{\boldsymbol{2\pi}}$ and therefore $f(z) = -i\dfrac{\Gamma}{2\pi}\ln z$ where Γ is the **circulation** (free vortex).libre)

If **Γ>0**, the vortex rotates in the **trigonometric diretion**.

If **Γ<0**, the vortex rotates **clockwise**.

I.10.4 Corners and stopping points

A "**stopping point" is** a point where the velocity is zero.

Consider the plane flow modeled by the complex velocity potential:

$f(z) = \mathbf{C}\, z^{m+1}$ where $m \geq -\frac{1}{2}$

In cylindrical coordinates : $z = r\,e^{i\theta}$ and then $f(z) = \mathbf{C}\, r^{m+1} e^{i(m+1)\theta}$

Then we have : $\begin{cases} \varphi(r,\theta) = \mathbf{C}\, r^{m+1}\cos\left[(m+1)\theta\right] \\ \Psi(r,\theta) = \mathbf{C}\, r^{m+1}\sin\left[(m+1)\theta\right] \end{cases}$

The velocity fields is obtained by : $\vec{V} = \begin{cases} v_r = \dfrac{\partial \varphi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \Psi}{\partial \theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta} = -\dfrac{\partial \Psi}{\partial r} \end{cases}$

We find :

$$\vec{V} = \begin{cases} v_r = \mathbf{C}(m+1)r^m \cos[(m+1)\theta] \\ v_\theta = -\mathbf{C}(m+1)r^m \sin[(m+1)\theta] \end{cases}$$

Note that $v_r = v_\theta = 0$ for $r = 0$ $\Rightarrow$ **the origin is the stopping point.**

The stream line passing through the stop point must therefore verify :

$\Psi(r,\theta) = \mathbf{C}^{te} = \Psi_{\mathbf{A}}$ where $\Psi_{\mathbf{A}} = \Psi(r_{\mathbf{A}}, \theta_{\mathbf{A}}) = \mathbf{C}\, r_{\mathbf{A}}{}^{m+1} \sin[(m+1)\theta_{\mathbf{A}}] = 0$

The equation for this current line is then written :

$$\mathbf{C}\, r^{m+1} \sin[(m+1)\theta] = 0 \Rightarrow \begin{cases} r = 0\ \forall\theta \\ \sin[(m+1)\theta] = 0\ \forall r \end{cases}$$ stop Point

$$\theta = \frac{n}{(m+1)}\pi\ \forall r \ \Leftarrow (m+1)\theta = n\pi\ \forall r$$

si n=0 : $\theta = 0\ \forall r \Rightarrow$ half - right Ax

Since stream lines can be likened to impassable barriers, those passing through the stopping point form "corners": these are the ***stopping corners***.

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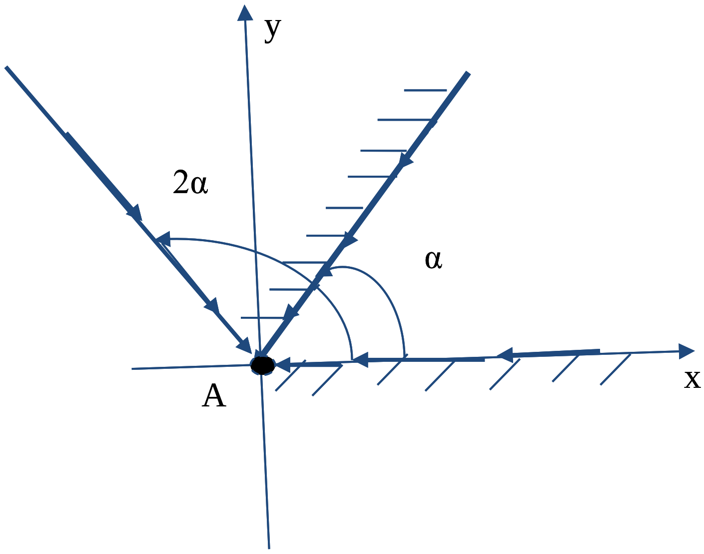


Let's now analyze the fluid flow between these stop wedges for a few specific values of m.

$$f(z) = C\, z^{m+1} \text{ où } m \geq -\frac{1}{2}$$

- **Case where m=1**

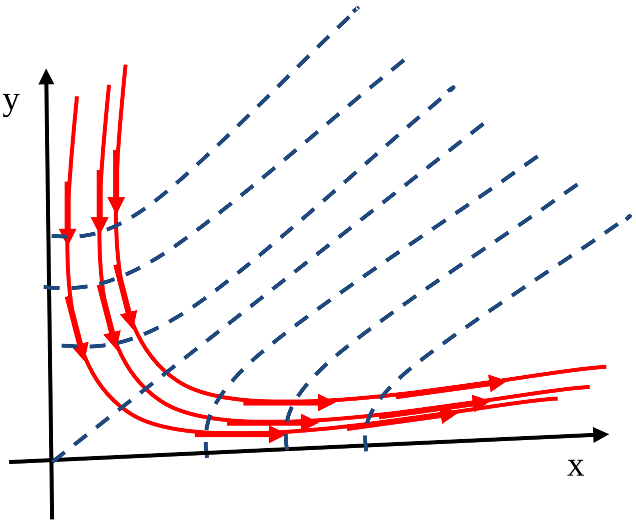
$$\Psi(r,\theta) = C\, r^2 \sin[2\theta] = C^{te} \quad \text{and} \quad \alpha = \frac{\pi}{m+1} = \frac{\pi}{2} \Rightarrow \textit{right angle corner}$$

$$\Rightarrow \quad \Psi(r,\theta) = 2C\, r^2 \sin\theta \cos\theta = 2C \underbrace{r \sin\theta}_{y} \underbrace{r \cos\theta}_{x} = C^{te}$$

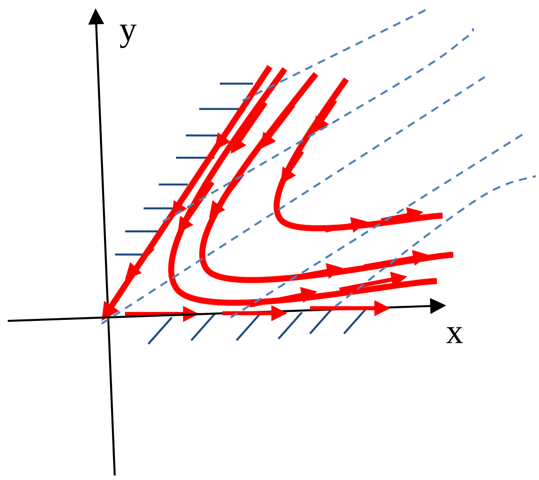
$$\Psi(r,\theta) = C^{te} \quad \Leftrightarrow \quad 2C\, xy = C^{te}$$

$y = \dfrac{C^{te}}{x}$ ***inside this corner, the current lines are hyperbolas***

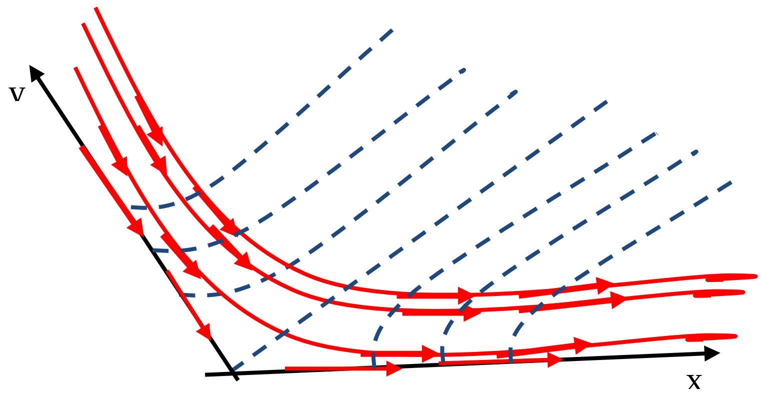
As equipotentials are⊥ at all points, they are also hyperbolas.



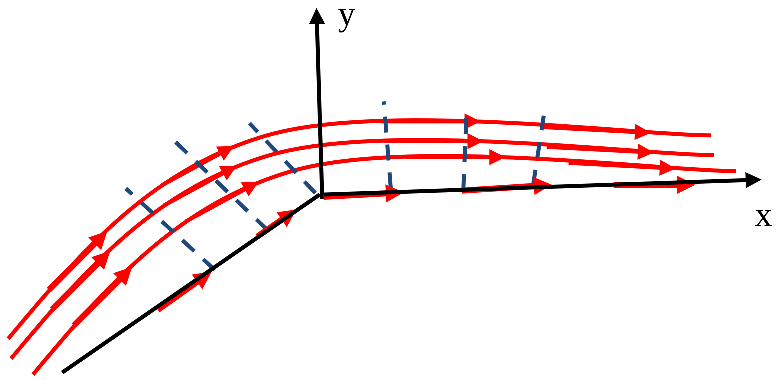
- **Case where m>1** $\alpha = \dfrac{\pi}{m+1} < \dfrac{\pi}{2}$



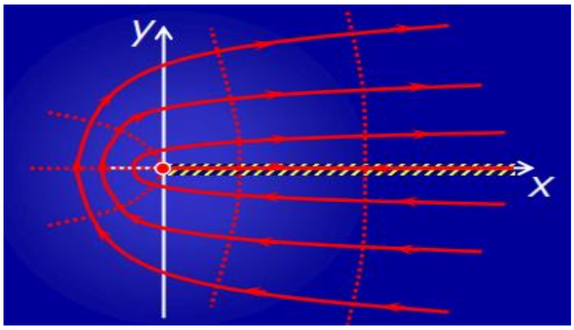
- **Case where 0<m<1** $\frac{\pi}{2} < \alpha = \frac{\pi}{m+1} < \pi$



- **Case where** $-\frac{1}{2}$**<m<0,** $\pi < \alpha = \frac{\pi}{m+1} < 2\pi$



➢ **Case** $m = -\frac{1}{2}$ $\alpha = 2\pi$



I.10.5 Doublet and dipôle

We know that for a flow to be described by a stream function and a velocity potential, both functions must satisfy *Laplace*'s equation :

$$\Delta\Psi = 0 \quad \text{Et} \ \Delta\varphi = 0 \ \Rightarrow \ f(z) = \varphi + i\Psi$$

Let's consider 2 flows such as:

$$\Delta\Psi_1 = 0 \ \text{et} \ \Delta\varphi_1 = 0 \Rightarrow \ f_1(z) = \varphi_1 + i\Psi_1$$

$$\Delta\Psi_2 = 0 \ \text{et} \ \Delta\varphi_2 = 0 \Rightarrow \ f_2(z) = \varphi_2 + i\Psi_2$$

Since *Laplace*'s equation is linear :

$$\Delta(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1\Delta\varphi_1 + \lambda_2\Delta\varphi_2 = 0$$

$$\Delta(\lambda_1\Psi_1 + \lambda_2\Psi_2) = \lambda_1\Delta\Psi_1 + \lambda_2\Delta\Psi_2 = 0$$

So if we put $\varphi = \lambda_1\varphi_1 + \lambda_2\varphi_2$ et $\Psi = \lambda_1\Psi_1 + \lambda_2\Psi_2$ then:

$$\Delta\Psi = 0 \ \text{Et} \ \Delta\varphi = 0 \Rightarrow f(z) = \varphi + i\Psi = \lambda_1 f_1(z) + \lambda_2 f_2(z)$$

Consequently, *f(z)* describes the flow resulting from the superposition of the two flows f_1 and f_2

Several elementary flows can therefore be superimposed to create more advanced flows, simply by adding the corresponding complex potentials.

I.10.6 Association of a source and a sink:

Let's consider a source with flow rate +*q*, located at *x*=*a*, onto which we superimpose a sink with flow rate -*q*, located at *x*=-*a*.

The resulting complex potential is written as:

$$f(z) = +\frac{q}{2\pi}\ln(z-a) - \frac{q}{2\pi}\ln(z+a) \quad \text{let's put: } \begin{cases} z_1 = z - a = r_1 e^{i\theta_1} \\ z_2 = z + a = r_2 e^{i\theta_2} \end{cases}$$

Hence :

$$f(z)=\frac{q}{2\pi}\left(\ln z_1-\ln z_2\right)=\frac{q}{2\pi}\left(\ln r_1+i\theta_1-\ln r_2-i\theta_2\right)$$

$$\Rightarrow \quad f(z)=\frac{q}{2\pi}\left[\ln\frac{r_1}{r_2}+i(\theta_1-\theta_2)\right]\Rightarrow \begin{cases}\varphi=\dfrac{q}{2\pi}\boldsymbol{ln}\dfrac{r_1}{r_2}\\ \boldsymbol{\Psi}=\dfrac{q}{2\pi}(\theta_1-\theta_2)\end{cases}$$

$$\begin{cases}\varphi=\dfrac{q}{2\pi}\ln\dfrac{r_1}{r_2}\\ \Psi=\dfrac{q}{2\pi}(\theta_1-\theta_2)\end{cases}$$ then, the stream lines are such as :

$$\Psi=\frac{q}{2\pi}(\theta_1-\theta_2)=\mathbf{C^{te}}$$

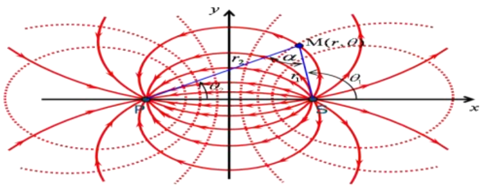


Figure 15 stream lines for a source and a sink

Let's extend the distance between the well and the source to 0.

$$f(z)=+\frac{q}{2\pi}\boldsymbol{ln}(z-a)-\frac{q}{2\pi}\boldsymbol{ln}(z+a)=\frac{q}{2\pi}\boldsymbol{ln}\left(\frac{z-a}{z+a}\right)=\frac{q}{2\pi}\boldsymbol{ln}\left(\frac{z(1-a/z)}{z(1+a/z)}\right)$$

$$f(z)=\frac{q}{2\pi}\boldsymbol{ln}\left(\frac{1-a/z}{1+a/z}\right) \quad \text{où} \quad \frac{1}{1+a/z}\xrightarrow[a\to 0]{}1-a/z$$

Then $f(z)\approx\frac{q}{2\pi}\boldsymbol{ln}\left[(1-a/z)^2\right]=\frac{q}{2\pi}2\,\boldsymbol{ln}(1-a/z)\approx\frac{q}{2\pi}2\left(-\frac{a}{z}\right)\approx\frac{q}{2\pi}2\left(-\frac{a}{z}\right)$

Let’s put $2aq = p$ be the dipôle moment : $f(z)=-\frac{1}{2\pi}\frac{p}{z}$

$$f(z)=-\frac{1}{2\pi}\frac{p}{z}=-\frac{1}{2\pi}\frac{p}{r\,\boldsymbol{e}^{i\theta}}=-\frac{1}{2\pi}\frac{p}{r}\boldsymbol{e}^{-i\theta}=-\frac{1}{2\pi}\frac{p}{r}\left(\boldsymbol{cos}\,\theta-i\,\boldsymbol{sin}\,\theta\right)=\varphi+i\boldsymbol{\Psi}$$

hence $\begin{cases}\varphi=-\dfrac{1}{2\pi}\dfrac{p}{r}\boldsymbol{cos}\,\theta\\ \boldsymbol{\Psi}=\dfrac{1}{2\pi}\dfrac{p}{r}\boldsymbol{sin}\,\theta\end{cases}$ $\quad \Psi=\mathbf{C^{te}} \iff \frac{1}{2\pi}\frac{p}{r}\sin\theta=\mathbf{C^{te}}$

stream line equation

$$\Rightarrow \quad \frac{1}{r}\sin\theta = C^{te} \Rightarrow \quad r\sin\theta = C^{te} r^2 \Rightarrow \quad y = C^{te}(x^2+y^2) \Rightarrow \quad y = C^{te}(x^2+y^2)$$

$$\Rightarrow \quad x^2+y^2-C^{te}y = 0 \Rightarrow x^2+y^2-Ky = 0 \Rightarrow \underbrace{x^2+(y-K/2)^2 = (K/2)^2}_{\text{equation of a circle with center (0,K/2) and radius K/2}}$$

$\Psi = C^{te} \quad \Leftrightarrow$

stream lines are circles all centered on the y axis, and all passing through the origin.

Flow generated by a **dipole** $\quad f(z) = -\frac{1}{2\pi}\frac{p}{z}$

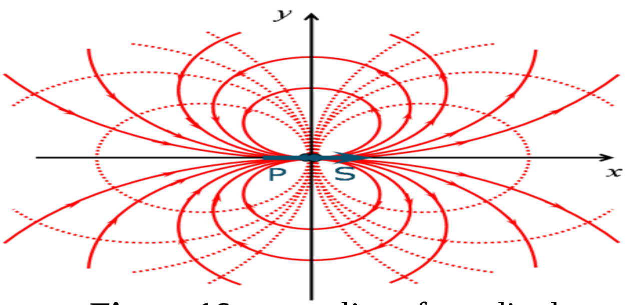


Figure 16 stream lines for a dipole

I.10.7 Uniform flow around a circular cylinder with circulation

Let's consider a uniform flow around a circle in the presence of a circulation Γ centered at the origin. The complex potential function is written: $f(z) = V_0\left(z + \frac{a^2}{z}\right) - i\frac{\Gamma}{2\pi}\ln z$

In view of the logarithmic singularity, the complex plane will be equipped with the half-axis cutoff

$$x \geq 0$$

The complex velocity of this flow is expressed as:

$V(z) = V_0\left(1 - \frac{a^2}{z^2}\right) - i\frac{\Gamma}{2\pi z}$ It cancels out at points with affixes z_A such that :

$$z_A^2 - i\frac{\Gamma}{2\pi V_0} z_A - a^2 = 0$$

This provides two stopping points:

$$z_A = z_{A'} = i\frac{\Gamma}{2\pi V_0} \pm \frac{1}{2}\sqrt{4a^2 - \frac{\Gamma^2}{4\pi^2 V_0^2}}$$

There are several cases depending on the descriminant.

- **Case 1** $\quad 0 \leq \Gamma \leq 4\pi a V_0$

The discriminant is then positive and the affixes of the two points have the same modulus:

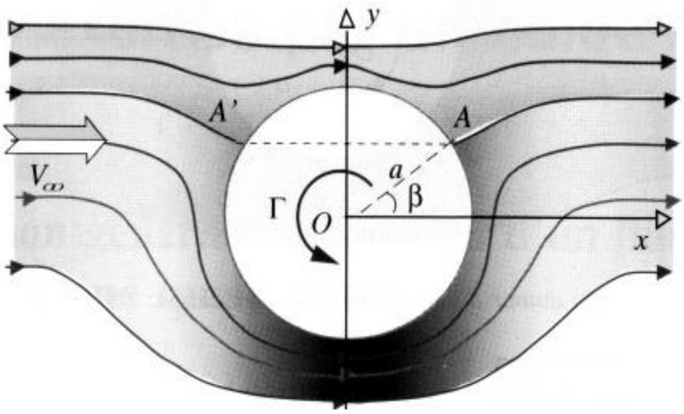


Figure 17 flow around a cylinder with circulation (low circulation)

$$\sqrt{x_A^2 - y_{A'}^2} = \sqrt{\frac{16\pi^2 a^2 V_0^2 - \Gamma^2}{16\pi^2 V_0^2} + \frac{\Gamma^2}{16\pi^2 V_0^2}} = a$$

The two stopping points are therefore on the circle of radius a, in symmetrical positions with respect to axis Oy. They are marked by the polar anglesβ and π-β respectively with:

$$\sin\beta = \frac{\Gamma}{4\pi a V_0}$$

The general flow configuration is shown in figure (I.17).

Without circulation, there are two stopping points at the intersection of the circle and the real axis. We can therefore see that the influence of traffic is equivalent to shifting the two stopping points symmetrically with respect to Oy by an ordinate proportional to the value of the traffic.

- **Case 2** $\Gamma = 4\pi a V_0$

For this critical circulation value, the two stop points merge with the intersection of the circle and the Oy axis($\beta=\pi/2$). This gives the configuration shown in figure (I.18).

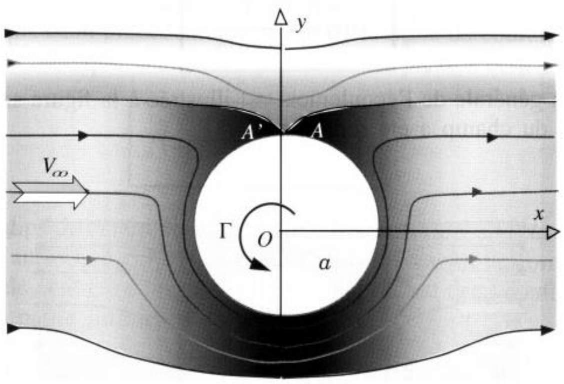


Figure 18 flow over cylinder with circulation (critic circulation)

- **Case 3** $\Gamma \succ 4\pi a V_0$

The discriminant of the equation of the affixes of the stopping points is then negative, so the roots take the form:

$$z_A = z_{A'} = i\left(\Gamma \pm \sqrt{\Gamma^2 - 4a^2\pi^2V_0^2}\right)/\,4\pi a V_0$$

These are pure imaginary, which means that the two stopping points are on the oy axis. The product of the roots is worth in modulus a^2 . This leads to the configuration shown in figure(I.19).

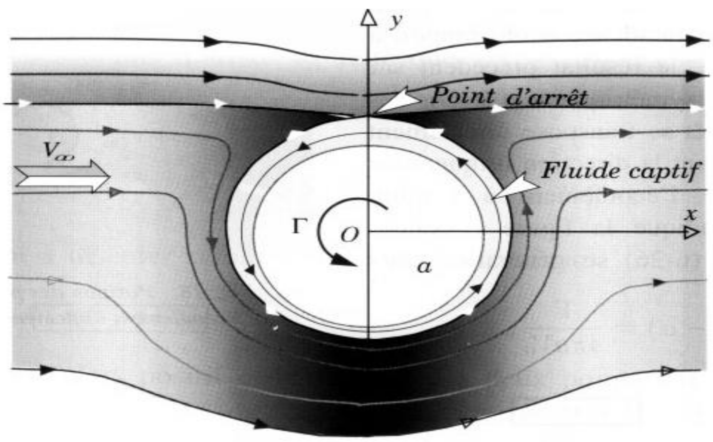


Figure 19 flow around cylinder with circulation (strong circulation)

I.11. Applications

Exercise 1 :

The velocity field of the two-dimensional flow is given by :

$\vec{V} = (3 + 2xy + 4t^2)\vec{i} + (xy^2 + 3t)\vec{j}$

Find the velocity and the acceleration at point (1,2) after 2s.

Exercise2:

The two-dimensional velocity vector field are given by : $\vec{V} = 2y^2\vec{i} + 3x\vec{j}$ In (x,y,z)=(2,2).

Calculate:

l /Local velocity

2/Local acceleration

3/ Convective acceleration

Exercise3:

Determine the expressions of the streamlines for the following velocity fields:

$1/\vec{V} = 3x\vec{i} + 6z\vec{k}$

$2/\vec{V} = 4z\vec{i} + 9x\vec{k}$

$3/\vec{V} = 2z\vec{i} + 3x\vec{k}$

$4/\vec{V} = 4y\vec{j} + 8z\vec{k}$

Exercise4:

The components of the velocity field of the two-dimensional, incompressible flow are given by the following equations: $\begin{cases} u = y^2 - x(1 + x) \\ v = y(2x + 1) \end{cases}$

Show that the flow is irrotational and satisfies the continuity equation.

Exercise 5:

The velocity distribution for a two-dimensional incompressible steady flow is given by :

$u = \dfrac{-x}{x^2 + y^2} \qquad v = \dfrac{-y}{x^2 + y^2}$

a- show that this distribution satisfies the continuity equation?

b- show that the flow satisfies Laplace's equation if the velocity field is derived from a potential.

Exercise 6:

The components of the velocity field of the two-dimensional incompressible flow are given by the following equations $\begin{cases} u = y^2 - x(1 + x) \\ v = y(2x + 1) \end{cases}$

Show that the flow is irrotational and satisfies the continuity equation.

Exercise 7:

The y-direction velocity component of a two-dimensional flow is given by: $v = 3xy + x^2y$

Determine the component of the velocity in the x direction that satisfies the continuity equation

Exercise 8:

A two-dimensional flow is defined by the velocity coordinates

U=-4m/s, v=-2m/s.

- Determine the corresponding stream function and velocity potential.
- Draw the equipotential line through the origin.

Exercise 9:

Determine the corresponding stream function for the following velocity potential:

φ =x3-3xy2

- Plot the stream function ψ=0, which passes through the origin.

Exercise 10:

For a two-dimensional flow, the velocity potential is given by : $\phi = x^2 - y^2$

- Determine the components of the velocity in the x and y directions.
- Show that the velocity satisfies the continuity and irrotationality conditions.
- Determine the stream function and the flow rate between the current functions (2,0) and (2,2).

Show that the stream functions and equipotentials form an orthogonal network perpendicular to the point (2,2).

Exercise 11:

Consider a plane flow modelled by the following complex potential velocity function :

$f(z) = K \ln z$

where z is the complex variable and K is a real constant.

1/ Write f(z) in complex form.

2/ Give the expression of the stream function and the velocity potential.

3/ Determine the components of the velocity(vr,vθ).

Exercise 12:

A two-dimensional fluid flow is described by the following stream function:

$\Psi = \left(\frac{U}{L}\right)xy$ where U and L are constants.

1. Show that this flow has a potential and deduce the components of the velocity.
2. Give the expression for the velocity potential.
3. Give the expression for the complex potential function.
4. Determine the stagnation points and the stream function passing through the stagnation point.

CHAPTER II

Integral Conservation Laws: Reynolds Transport Theorem (RTT)

II. Integral Conservation Laws

The integral conservation laws are the fundamental principles in physics and engineering that describe the conservation of certain physical quantities, such as mass, momentum and energy within a system or control volume. These laws are expressed in the form of integral equations that consider the variations of these quantities within a volume as well as the flows across these boundaries.

The Reynolds Transport Theorem (RTT) is an essential mathematical tool that relates the time derivative of a volume integral to surface dynamic integrals, thus facilitating volumes that may be mobile, fixed or deformable the analysis of dynamic systems. It is particularly useful when working with control.

II.1 Control volume

A control volume, often referred to as V(C) is a fixed or moving region of space chosen for the analysis of a fluid flow. This volume can be real or imaginary, and its size, shape and position are defined to study the phenomena occurring within it. The control volume allows the study to be focused on a subset of the space where variables such as mass, velocity, temperature or energy are measured and analyzed. It is an imaginary volume through which fluid can flow. The focus is on the physical quantities passing through the surface.

There are two types of control volume:

- **Fixed:** The volume remains stationary in space, and the fluid passing through its boundaries is observed. This type is common in industrial applications, for example in turbines or fluid pipes.
- **Mobile:** The volume moves with the fluid, and the boundaries of the volume change over time. This approach is often used to study the behavior of specific fluid particles.

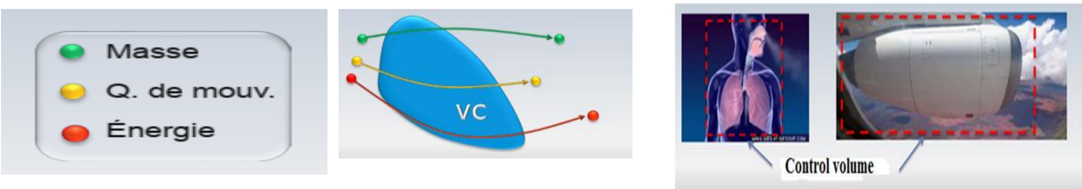
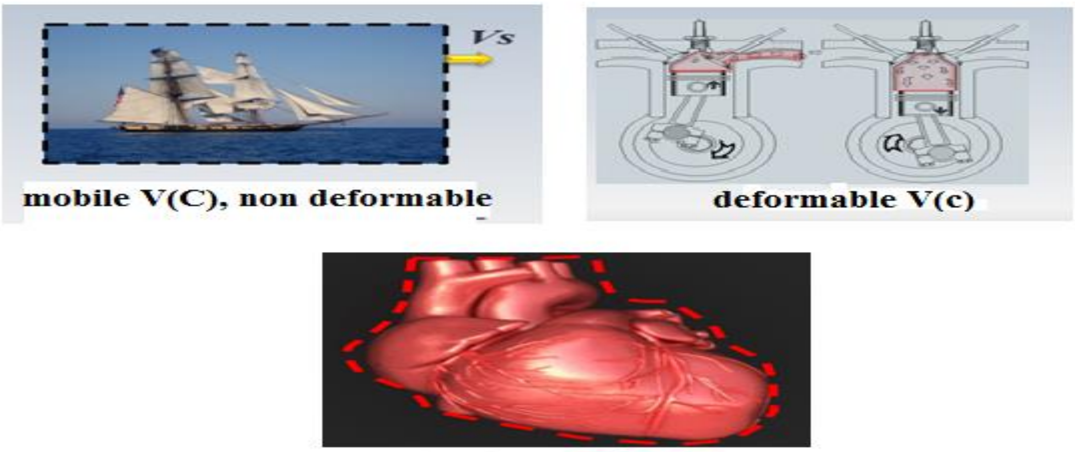


Figure 1 control volume



deformable V(C)

Figure 2 Control volume : mobile, deformable and non-deformable

II.2 Control Surface

The control surface, often referred to as the S(C) is the boundary of the control volume. It completely surrounds the control volume and delimits the region where flows (such as those of mass, momentum or energy) enter and leave. It plays a key role because conservation laws often involve surface integrals calculated on this boundary.

The control volume and its control surface allow conservation equations to be applied in a simplified way:

- **Conservation of Mass**: By analysing the inflow and outflow of mass through the control surface, the changes in mass in the control volume can be calculated.
- **Conservation of momentum**: Using the flow of momentum through the control surface, we can determine the forces acting on the control volume.
- **Conservation of Energy**: The energy exchanged across the control surface helps to calculate the internal energy variations of the volume.

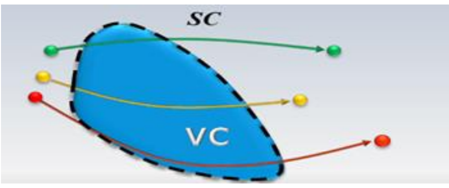


Figure 3 Control surface

II.3 Material volume

A material volume V(M) is a region of the fluid that always contains the same fluid particles over time. Unlike the control volume, the material volume moves and deforms with the fluid flow, following the particles as they move through the velocity field. It is also referred to as a 'set of particles' or a 'slice of fluid' that is followed as it moves.

Characteristics of a material volume:

- Consisting of a fixed group of particles: The material volume always retains the same fluid particles within it, regardless of the deformations or movements undergone by the fluid.
- Deformation and movement: The material volume changes shape and position according to variations in velocity and the internal forces of the fluid.
- Lagrangian approach: This approach makes it possible to study the specific behaviour of a slice of fluid, which is useful for analysing the internal changes (e.g. deformation or internal energy) of fluid particles.

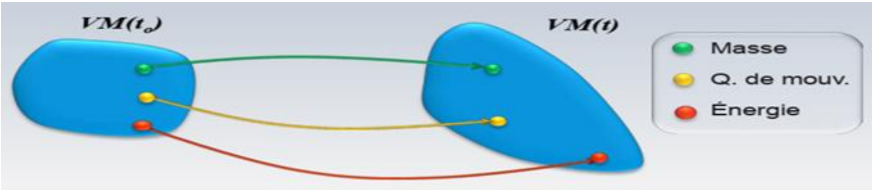


Figure 4 Material volume

The closed system *V(M)* is associated with Lagrangian kinematics, while the open system *V(C)* with the Eulerian approach.

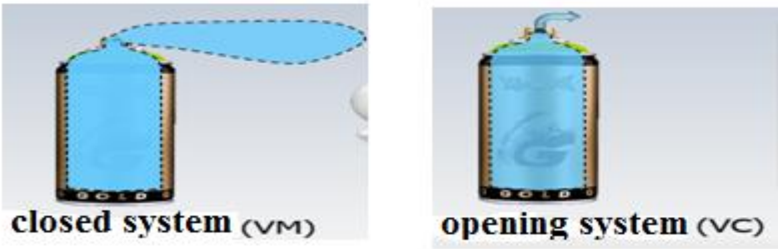


Figure 5 Closed system and opening system

II.4 Material surface

The material surface S(M) is the boundary of a material volume and is a closed surface always containing the same fluid particles on its boundary. Like the material volume, the material surface moves and deforms with the fluid particles, maintaining the initial particles present at the boundary.

Characteristics of the material surface:

- Composed of the same particles over time: The particles that are on the material surface at the start remain on that surface during movement.
- Deformation as a function of forces: The material surface undergoes the same deformations as the material volume, because it is subjected to the tensile, compressive or shear forces of the fluid.
- Zero flow through the surface: By definition, no fluid particle flow passes through the material surface, because the surface contains a fixed set of particles. This differs from open control surfaces, where fluid can flow in or out.

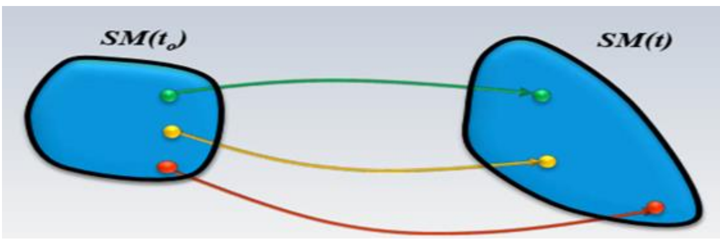


Figure 6 Material surface

The material volume and **material surface** allow a tracking approach for particles in a fluid flow, making it possible to analyse the transformations undergone by the particles themselves. They are therefore complementary to the concepts of control volume and control surface, which remain fixed in space and enable a more global integral analysis of fluid systems.

II.5 Examples:

1. Material Volume

Suppose we are studying the dispersion of a drop of ink in a moving container of water. If we choose a material volume that initially contains only the ink molecules, this volume will follow these specific molecules as they disperse in the water.

- Material volume: The volume containing the ink will expand and deform over time as a result of movement and mixing in the water.

- Study of internal properties: By following this material volume, we can observe how the ink concentration changes, how it is diluted, and how diffusion acts on the particles in the surrounding fluid.

2. Material Surface

Imagine an air bubble in a glass of water. The material surface of this bubble is the boundary between the air inside the bubble and the water outside.

- Material surface: The surface of the bubble always contains the same air particles on its boundary, even if the bubble rises into the water or changes shape due to pressure forces.

- Deformation analysis: This surface can be used to study the effects of water pressure on the shape of the bubble and observe how it deforms under the forces of tension and compression.

II.6 Reynolds transport theorem (RTT)

Before looking at the **Reynolds Transport Theorem,** we need to introduce a few basic concepts.

II.6.1 Flow concept

To measure the quantity of matter passing through a surface (S) per unit of time and surface area, we introduce the notion of flow: flow of mass, momentum, energy, etc.

- **Volume and mass flow**

The elementary volume flow dq_v through a surface dS is the volume of fluid dv that passes through this surface in a time interval dt, i.e.:

$$q_v = \oiint_S \vec{V}.\vec{n}ds \quad q_v = \frac{dv}{dt} = \vec{V}.d\vec{s} = \vec{V}.\vec{n}.ds$$

Then, the total volum flow q_v over a surface *S* is :

$$q_v = \int \vec{V}.\vec{n}.ds \tag{II.1}$$

Likewise for mass flow: $q_m = \rho q_v$ or again

$$q_m = \frac{dm}{dt} = \int \rho\vec{V}.\vec{n}.ds$$

then:

$$q_m = \int \rho\vec{V}.\vec{n}.ds \tag{II.2}$$

II.6.2 Intensive quantities and extensive quantities

1. Intensive Quantities

An intensive quantity is a physical property that does not depend on the size or amount of the system or substance. These quantities remain constant regardless of how much material is present in the system.

Characteristics:

- **Scale-independent**: Dividing the system into smaller parts does not change the value of an intensive property.
- **Examples**:
  - Temperature (T)
  - Pressure (P)
  - Density (ρ)
  - Specific heat (Cp,Cv)
  - Specific volume (v)

2. Extensive Quantities

An extensive quantity is a physical property that depends on the size, amount, or extent of the system or substance. These quantities scale with the system's size or volume.

Characteristics:

- **Scale-dependent**: Dividing the system into smaller parts reduces the value of the extensive property proportionally.
- **Examples**:

- Mass (m)
- Volume (V)
- Energy (E)
- Enthalpy (H)
- Entropy (S)

II.6.3 Formulation of the Reynolds Transport Theorem, RTT

The intensive quantities, are independent of the mass of the system. In particular, we'll be looking at quantities associated with **mass m, momentum mV and energy E.**

To do this, we reduce each of the properties (**m, mV, E)** by the mass **m**, to obtain the intensive quantities (1, V, e).

These relationships can be generalized for any extensive quantity **B** with a corresponding intensive quantity, i.e. per unit mass, **b = B /m**:

$$B=\int_{v}\rho b dv \qquad \text{(II.3)}$$

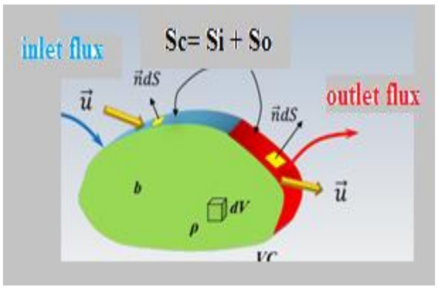
1.The net flow of B

Consider a volume V(C) (fixed) bounded by S(C), through which flows a fluid carrying B.The net flow of B through the control surface S(C) can therefore be written as:

$$\dot{B}=\frac{dB}{dt}=\int_{So}\rho b\vec{V}ds-\int_{Si}\rho b\vec{V}ds \qquad \text{(II.4)}$$

$$\dot{B}_{net=}\int_{Sc}\rho b\vec{V}ds \qquad \text{(II.5)}$$

With b=B/m (II.6)



The flow rate through the surface of a control volume corresponds to the quantity of B that ‘accumulates’ (negative or positive) per unit of time in the control volume. This variation in B in the control volume can be written as:

$$\left(\frac{dB}{dt}\right)_{Vc} = \frac{d}{dt}\int_{Vc} \rho b dv \qquad (II.7)$$

Accumulation over time in the control volume.

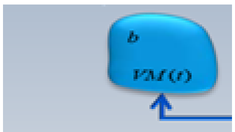
2. Balance of a control volume

For a control volume, in the absence of sources (sinks), we recognize the following principle:

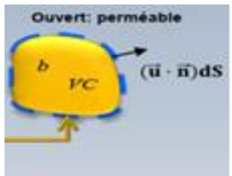
Accumulation in V(c) + **balance of flows**(through S(c)=0

In mathematical form, we write the above principle as.

closed system



$$\frac{dB}{dt} = \frac{d}{dt}\int_{Vc} \rho b dv + \int_{Sc} \rho b(\vec{V}.\vec{n})ds = 0 \qquad (II.8)$$



Time variation of **B** when following the system	=	Time variation of **B** in the control volume	+	Net flow of **B** through surface Sc of volume Vc

This relationship was presented for a fixed control volume.

$$\vec{V}_{rel} = \vec{V}_{fluid} - \vec{V}_s \qquad (II.9)$$

If the control volume deforms, consider the relative velocity between the velocity V of the fluid and that Vs of the volume Vc

- b is the intensive property, $b = \frac{B}{m}$
- ρ is the density,
- $\vec{V}$ is the velocity field,
- $\vec{n}$ is the unit normal vector at the control surface (CS),
- Vc is the control volume,
- Sc is the control surface.

B		**b=B/m**
Mass	m	1
Momentum	mV	V
Energie	E	e

Vs=0 if Vc is fixed

B: Extensive property. A quantity in the closed system

b: Intensive property. Property B per unit mass

ρ: density of the fluid

V: velocity of the fluid

$\vec{V}_{rel} = \vec{V}$: if the surface of the control volume is fixed

$\vec{V}_{rel} = \vec{V} - \vec{V}_s$: if the surface of the control volume is moving with velocity

dS : aire élémentaire sur la surface de contrôle, Sc

$\vec{n}$: outward unit normal of the elementary SC dS

dv: volume element in the Vc

II.7 Conservation of mass (continuity equation)

II.7.1 Application of RTT for the mass

In this case, let's analyses the conservation of mass with B=m and b=B/m=1.

Even if the material volume deforms, the mass in it remains the same over time.

$$\left.\frac{dB}{dt}\right|_{system} = \left.\frac{dm}{dt}\right|_{system} = q_m = 0 \qquad \text{(II.10)}$$

(The mass of a system remains constant over time)

Since there is no accumulation (or loss) of mass takes place in the control volume. The sum of positive and negative flows (volumes) is zero.

The conservation of mass equation takes the form:

$$\frac{d}{dt}\int_{Vc} \rho dv + \int_{Sc} \rho(\vec{V}.\vec{n})ds = 0 \qquad \text{(II.11)}$$

We have $\int_{Sc} \rho(\vec{V}.\vec{n})ds = \sum q_m$ sum of mass flow. (II.12)

If the control volume is fixed, $\vec{V}_s = 0$ and $\vec{V}_{rel} = \vec{V}$ (the flow velocity) and the flow is **steady-state, then:

$$\frac{d}{dt}\int_{Vc} \rho dv + \int_{Sc} \rho(\vec{V}.\vec{n})ds = 0$$

$$\int_{Sc} \rho(\vec{V}.\vec{n})ds = 0 \qquad \text{(II.13)}$$

$$\frac{d}{dt}\int_{Vc}\rho dv = 0 \qquad (II.14)$$

If the fluid is incompressible ρ=cte, then we have, even in unsteady conditions,

In the case of uniform inputs and outputs (1D), the previous equations become:

$$\int_{Sc}\rho(\vec{V}.\vec{n})ds = 0 \Rightarrow \sum_{outlets}\rho_i V_i S_i = \sum_{inlets}\rho_j V_j S_j \Rightarrow \sum_{outlet} qm_i = \sum_{inlets} qm_j \text{ (Mass flow rates)} \qquad (II.15)$$

$$\int_{Sc}(\vec{V}.\vec{n})ds = 0 \Rightarrow \sum_{outlets} V_i S_i = \sum_{inlets} V_j S_j \Rightarrow \sum_{outlet} qv_i = \sum_{inlet} qv_j \text{ (volume flows)} \qquad (II.16)$$

In practice, we often find applications with a single inlet and a single outlet, such as a pipe carrying water, or a passage for ventilation in a building. These types of problems are modelled using the notion of a flow tube.

This is a conceptually fictitious pipe (which can sometimes be correspond to a physical tube) with an inlet cross-section Ai, an outlet cross-section Ao , both plane, and side walls Ap tangent to the velocity vector.

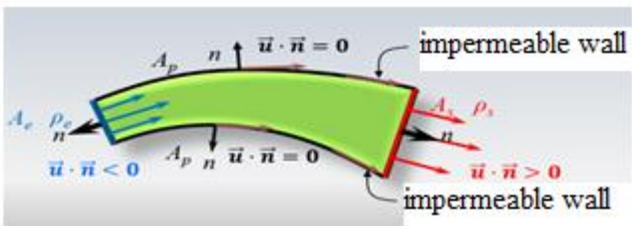


Figure 7 Example of current tube

Given that at side walls $\vec{V}.\vec{n}=0$, the surface integral only needs to consider the inlet and outlet, i.e. :

$$\int_{Sc}\rho(\vec{V}.\vec{n})dA = \int_{Si}\rho(\vec{V}.\vec{n})dA + \int_{So}\rho(\vec{V}.\vec{n})dA + \underbrace{\int_{Sw}\rho(\vec{V}.\vec{n})dA}_{=0}$$

In incompressible conditions (ρ=cste) and if the velocities u_e and u_s are considered to be uniform, then:

$$\int_{Si}(\vec{V}.\vec{n})dA = -u_i A_i = Qv_i \quad \text{Inlets}$$

$$\int_{So}(\vec{V}.\vec{n})dA = u_o A_o = Qv_o \quad \text{Outlets}$$

$$\Rightarrow u_o A_o = u_i A_i \Rightarrow Qv_i = Qv_o \text{ (Constant volume flows)}$$

When**ρ≠ cste,** we have **$\rho_i u_i A_i = \rho_o u_o A_o$**

II.8 Conservation of momentum

II.8.1 Application of RTT for momentum

For any extensive property B, the RTT states:

$$\left.\frac{dB}{dt}\right|_{system} = \frac{d}{dt}\int_{Vc}\rho b dv + \int_{Sc}\rho b(\vec{V}_{rel}.\vec{n})ds \qquad \text{with} \quad \vec{V}_{rel} = \vec{V}$$

For momentum $B = m\vec{V}$ $B = m\vec{V}$ and then $b = \frac{B}{m} = \vec{V}$ the RTT gives:

$$\left.\frac{d(m\vec{V})}{dt}\right|_{system} = \frac{d}{dt}\int_{Vc}\rho\vec{V}dv + \int_{Sc}\rho\vec{V}(\vec{V}.\vec{n})ds \tag{II.17}$$

$$\left.\frac{dB}{dt}\right|_{system} = \left.\frac{d(m\vec{V})}{dt}\right|_{system} = \sum\vec{F}_{syst} \quad \Rightarrow \sum\vec{F}_{syst} = \frac{d}{dt}\int_{Vc}\rho\vec{V}dv + \int_{Sc}\rho\vec{V}(\vec{V}.\vec{n})ds \tag{II.18}$$

(since at an instant t, the (moving) material volume coincides with the (fixed) control volume, we have the expression):

$$\sum\vec{F}_{VC} = \frac{d}{dt}\int_{Vc}\rho\vec{V}dv + \int_{Sc}\rho\vec{V}(\vec{V}.\vec{n})ds \tag{II.19}$$

Forces exerted on the VC

Variation in momentum in the VC

Flow of momentum through the SC

**Note that the forces on the control volume are sources (+) or sinks (-) of momentum.

A source (force experienced by the fluid) corresponds to an increase in its momentum.

A sink (force exerted by the fluid) corresponds to a decrease in its momentum.

The momentum equation is a vector equation, so it can be written for the 3 velocity components u, v and w, or in index notation for Vi with i=1,2,3.

$$\sum\vec{F}_{VC} = \frac{d}{dt}\int_{Vc}\rho\vec{V}dv + \int_{Sc}\rho\vec{V}(\vec{V}_{rel}.\vec{n})ds \tag{II.20}$$

$$\sum\vec{F}_{syst} = \frac{d}{dt}\int_{Vc}\rho\vec{V}_i dv + \int_{Sc}\rho\vec{V}_i(\vec{V}_{rel}.\vec{n})ds \tag{II.21}$$

Where Vi are the u, v and w components

Fi are Fx, Fy, Fz

If the control volume does not deform, $\vec{V}_{rel} = \vec{V}$ and if the flow is permanent, we have:

$$\sum \vec{F}_{VC} = \frac{d}{dt}\int_{Vc} \rho \vec{V} dv + \int_{Sc} \rho \vec{V}(\vec{V}_{rel}.\vec{n}) ds \qquad (II.22)$$

If, in addition, the inputs and outputs have uniform speeds, the integral over the Sc is replaced by the balance of the incoming and outgoing flows:

$$\int_{Sc} \rho \vec{V}(\vec{V}.\vec{n}) ds = \vec{V} \int_{SC} \underbrace{\rho \vec{V}.\vec{n} ds}_{qm} \text{ then } \sum \vec{F}_{VC} = \sum_{outlets} qm_i \vec{V}_i - \sum_{inlets} qm_j \vec{V}_j \qquad (II.23)$$

Note: The summations over i and j correspond to the number of inlets/outlets

In the case of a single input and output we have:

$$\sum \vec{F}_{VC} = qm(\vec{V}_{outlet} - \vec{V}_{inlet}) \qquad (II.24)$$

We can use the current tube as an example.

$$\sum \vec{F}_{VC} = (\rho_2 V_2 A_2)\vec{V}_2 - (\rho_1 V_1 A_1)\vec{V}_1 \qquad (II.25)$$

$$\sum \vec{F}_{VC} = \vec{P} + \vec{F}p_1 + \vec{F}p_2 + \vec{F}pp + \vec{F}rp \qquad (II.26)$$

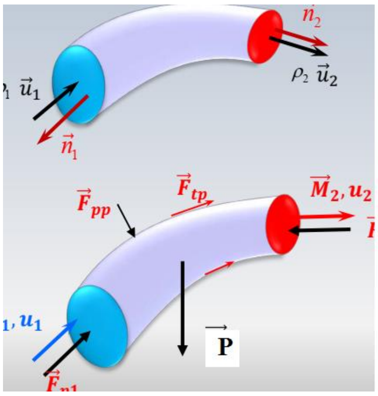


Figure 8 Momentum of current tube

II.9 Conservation of energy
II.9.1 Application of RTT for Energy

For the energy equation, it expresses the conservation of energy for a control volume, relating it to the fluxes across its boundaries and sources within the volume.

For any extensive property B, the RTT states:

$$\left.\frac{dB}{dt}\right|_{system} = \frac{d}{dt}\int_{Vc}\rho b dv + \int_{Sc}\rho b(\vec{V}_{rel}.\vec{n})ds$$

b is the intensive property : $b = \frac{B}{m}$**,** B=e then b=e/m=E

The extensive quantity b is then the total energy of the system, only mechanical and thermal energy exchanges are taken into account.

For the energy equation, the conserved extensive property is the **total energy** (E), which includes internal energy (U), kinetic energy (KE), and potential energy (PE):

$$E = U + KE + PE = \int_{VC}\rho\left(u + \frac{V^2}{2} + gz\right)dv \qquad \text{(II.27)}$$

Applying RTT,

$$\left.\frac{de}{dt}\right|_{system} = \frac{d}{dt}\int_{Vc}\rho e dv + \int_{Sc}\rho e(\vec{V}.\vec{n})ds \qquad \text{(II.28)}$$

Where $u + \frac{V^2}{2} + gz$ is the specific total energy (per unit mass).

This equation represents the **rate of change of total energy** in the system.

Including Heat Transfer and Work

The energy equation must also account for heat transfer ($\dot{Q}$) and work done ($\dot{W}$), such as shaft work or boundary work:

$$\frac{d}{dt}\int_{Vc}\rho e dv + \int_{Sc}\rho e(\vec{V}.\vec{n})ds = \dot{Q} - \dot{W}_{out} \qquad \text{(II.29)}$$

Here:

$\dot{Q}$ is the heat added to the system.

$\dot{W}_{out}$ is the work done by the system.

Expanded Energy Equation in Terms of Fluxes

$$\frac{d}{dt}\int_{Vc}\rho e dv + \int_{Sc}\rho e(\vec{V}.\vec{n})ds = \int_{VC}\dot{q}dv - \int_{SC}\vec{\tau}.\vec{V}ds \qquad \text{(II.30)}$$

where:

- $\dot{q}$ represents volumetric heat generation,
- $\dot{\tau}$ is the stress tensor accounting for work contributions.

The RTT for the energy equation bridges the system and control volume perspectives, allowing energy conservation to be expressed in terms of temporal changes within a control volume, fluxes across boundaries, and external energy sources or sinks. It is fundamental in solving engineering problems involving heat transfer and fluid flow.

II.10 Applications

Exercise1:

a continuous flow of water in a tank containing several inlets and outlets (fig1).
1/ Specify the control volume.
2/ Determine the velocity in section3.
We give A_1=0.05m^2, A_2=0.01m^2, A_3=0.06m^2

$\vec{V}_1 = 4\vec{i}\,(m/s)$

$\vec{V}_2 = -8\vec{j}\,(m/s)$

$\vec{V}_3 = ?$

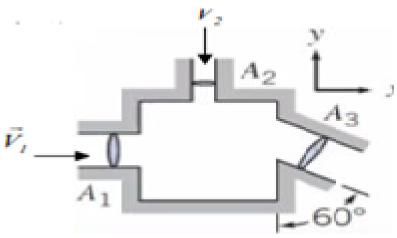


fig1

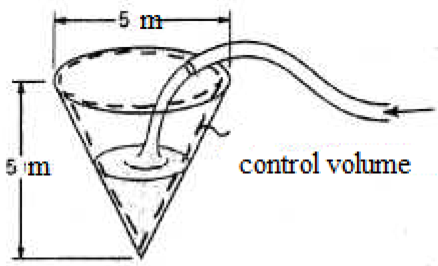
Exercise2:

Under normal conditions, air enters a compressor at a flow rate of 0.3m3/s (fig2) and exits the tank through a section with a diameter of 3cm and a density of 0.0511kg/m^3. Determine the rate of change with respect to time of the density in the tank.

fig(2)

Exercise3:

Estimate the filling time (in minutes) of the cone-shaped tank (fig3) with a height of 5m, a top diameter of 5m and a flow rate of Qv=2.67m^3/min.



fig(3)

Exercise4:

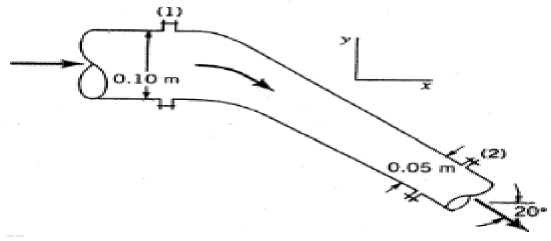
A channel of width 3m is provided with an inlet of uniform velocity V and an outlet whose velocity distribution is given by

u=4y-2y fig(4). Determine the velocity V at the inlet.

fig(4)

Exercise5:

The water flows through a 20° elbow with a flow rate of 0.025m3/s (fig5). The effects of viscosity and gravity are assumed to be negligible and the pressures in sections (1) and (2) are P1=150KPa and P2=14.5KPa respectively. Determine the components of the force required to hold the elbow in place.



(fig5)

Chapter III

Dimensional analysis and similarities

III.1 Introduction

Dimensional analysis is a practical method for checking the homogeneity of a physical formula through its dimensional equations, i.e. the decomposition of the physical quantities it involves into a product of basic quantities: length, duration, mass, electrical intensity, etc., all of which are irreducible.

Dimensional analysis is used to:

- Determine the unit of a quantity
- Check the homogeneity of a formula
- Predict the form of a physical law in order to find the solution to certain problems without having to solve an equation: for many of the physical phenomena studied, we can express a characteristic quantity of the phenomenon and deduce an order of magnitude.

Dimensional analysis can be applied in almost all areas of engineering. It is also a very useful additional tool in modern fluid mechanics. It is based on the principle of dimensional homogeneity and uses the dimensions of the relevant variables affecting the phenomenon in question.

III.2 Dimensions

The various physical quantities used in fluid mechanics can be expressed in terms of fundamental or primary quantities.

In the International System, the primary or fundamental physical quantities are mass, length, time and sometimes temperature (compressible flows) and are designated respectively by the letters M,L,T,θ. Quantities that are expressed as a function of the fundamental quantities are called secondary or derived quantities (speed, area, acceleration....). The expression of a derived quantity as a function of the fundamental quantity is called the Dimension of the physical quantity.

A quantity can be expressed dimensionally as M,L,T or F,L,T.

Example:

Flow = velocity x area= $\frac{L}{T}.L^2 = \frac{L^3}{T} = L^3.T^{-1}$

Kinematic viscosity υ=μ/ρ we have $\tau = \mu\frac{du}{dy}$ and $\mu = \frac{\tau}{\frac{du}{dy}} = \frac{\text{contra int e}}{\frac{L}{T}x\frac{1}{L}} = \frac{\text{force / aire}}{\frac{1}{T}}$

$$\frac{massexacceleration}{airex\frac{1}{T}} = \frac{Mx\frac{L}{T^2}}{L^2x\frac{1}{T}} = \frac{ML}{L^2T^2x\frac{1}{T}} = \frac{M}{LT} = ML^{-1}T^{-1}$$

$$\rho = \frac{mass}{volume} = \frac{M}{L^3} = ML^{-3}$$

Then the kinematic viscosity, $\nu = \frac{\mu}{\rho} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = ML^{-2}T^{-2}$

II.3. Principle of dimensional homogeneity

An equation is considered to be dimensionally homogeneous if the form of the equation does not depend on the units of measurement, or if the two terms of the equation have the same dimensions.

III.4 Dimensional analysis method

The use of dimensions enables us to determine whether a literal expression is homogeneous or not. This makes it possible to identify any errors. But dimensional analysis can also be used to find or guess physical laws when the theoretical solution is too complex.

When the system under study is too complex to allow a complete resolution of the fundamental equations, or when its behaviour is chaotic, dimensional analysis provides simple access to relationships between the various quantities characterising the system.

By grouping these different quantities into dimensionless numbers, it is also possible to establish similarities between the behaviour of similar but different systems (prototype/model).

The application of dimensional analysis to a practical problem is based on the assumption that certain variables affecting the phenomenon are independent. The number of variables characterising the problem is equal to the number of independent variables plus one. One is the number of dependent variables.

Dimensional analysis is used to obtain a functional relationship between the dependent variables and the independent variables.

The first step in dimensional analysis is to determine the variables involved in the problem. Naming these variables requires a good understanding of the phenomenon. The second step is to form the adimensional groups of these variables.

The Vachy-Buckingham -π method (π Buckingham theorem) is the most commonly used method in dimensional analysis.

Let's take the example of determining the regular head losses in a cylindrical pipe:
Les différentes grandeurs qui interviennent sont :

$\frac{\Delta P_t}{L}$ La perte de charge par unité de longueur,

D Le diamètre de la conduite,

ε La rugosité de la conduite,

v La vitesse moyenne de l'écoulement (ou le débit),

μ La viscosité du fluide,

ρ La masse volumique du fluide.

Par conséquent, il existe une relation entre ces différentes grandeurs :

$$\frac{\Delta P_t}{L} = f(D, \varepsilon, v, \mu, \rho)$$

The function f can be difficult to find, so dimensional analysis will enable us to establish a simpler relationship between a smaller number of dimensionless quantities.
A systematic method will enable us to find 3 dimensionless numbers:

$$\pi_1 = \frac{\Delta p_t}{L}\frac{D}{\rho V^2} = \lambda \qquad \pi_2 = \frac{\rho V D}{\mu} = \mathrm{Re} \qquad \pi_3 = \frac{\varepsilon}{D} = \varepsilon_r$$

This will enable us to establish $\pi_1 = \Phi(\pi_2, \pi_3)$

$$\Rightarrow \frac{\Delta p_t}{L}\frac{D}{\rho V^2} = \Phi\left(\frac{\rho V D}{\mu}, \frac{\varepsilon}{D}\right) \Rightarrow \frac{\Delta p_t}{L} = \frac{\rho V^2}{D}\Phi(\mathrm{Re}, \varepsilon_r)$$

The dimensional analysis shows that the regular head loss depends solely on the

The dimensional analysis shows that the regular head loss depends solely on the Reynolds number and the relative roughness of the pipe.

III.5. Vachy-Buckingham *π*-theorem

The π-Buckingham method expresses the resulting equation in terms of non-dimensional groups (π-terms). According to this theorem, if a phenomenon drives p variables: a1,a2,a3,.....,.ap such that one variable a1depends on the other independent variables a2,a3,.....,.an, the general functional relationship between the dependent variables and the independent variables can be expressed as follows:

a1=f(a2,a3,..................ap) (III.1)

Expression (III.1) can be written mathematically as:

□(a2,a3,.........ap)=0 (III.2)

That is, if an equation with p variables is homogeneous, it can be reduced to a relation between (p-q) dimensionless independent products, where q is the minimum number of dimensions required to describe the p variables, and we write:

f(π1, π2, π3,............................... πp-q)=0 (III.3)

In problems where all fundamental dimensions are considered, it is recommended to select the repeated variables using the following guidelines:

- Select the first variable from those describing the flow geometry.
- Select the second repeated variable representing the fluid properties.
- Select the third variable repeated with those characterising the movement of the fluid.

To illustrate this statement, let's go back to the previous example:

We had **p=6** variables (which require a minimum of **q=3** dimensions (M,L,T).

$$\left[\frac{\Delta p_t}{L}\right] = ML^{-2}T^{-2},\ [D] = L,\ [\varepsilon] = L,\ [V] = LT^{-1},\ [\mu] = ML^{-1}T^{-1},\ [\rho] = ML^{-3}$$

Consequently, the equation linking the 6 variables can be reduced to an equation linking **p-q = 3** dimensionless products:

$$\pi_1 = \frac{\Delta p_t}{L}\frac{D}{\rho V^2} = \lambda,\ \pi_2 = \frac{\rho V D}{\mu} = Re,\ \pi_3 = \frac{\varepsilon}{D} = \varepsilon_r$$

Buckingham's π theorem therefore allows the passage:

$$\frac{\Delta p_t}{L} = f(D, \varepsilon, V, \mu, \rho) \qquad \pi_1 = \Phi(\pi_2, \pi_3)$$

In order to apply this theorem, we need to use a systematic method:

- List the variables in the problem $\Rightarrow$**p**
- Write the equation in dimensions for each of the **p** variables
- Determine **q**, and therefore $\Rightarrow$**p-q** the number of dimensionless products characterising the problem.
- From the **p** variables, choose a number **q** that are dimensionally independent $\Rightarrow$q primary variables
- Form the **p-q** products π by combining the **p-q** non-primary variables with the **q** primaries so as to obtain dimensionless quantities.
- Formulate the relationship between the **p-q** products π found.

We will apply the method to the example of flow around a vertical plate to write the drag force exerted by the flow on this plate in dimensionless form.

The drag force is the force exerted by a flow on an object in the direction parallel to the flow. We will study the case of a rectangular flat plate.

The variables of the problem are: $F,h,L,\mu,\rho \Rightarrow \boldsymbol{p = 6}$

F: drag force

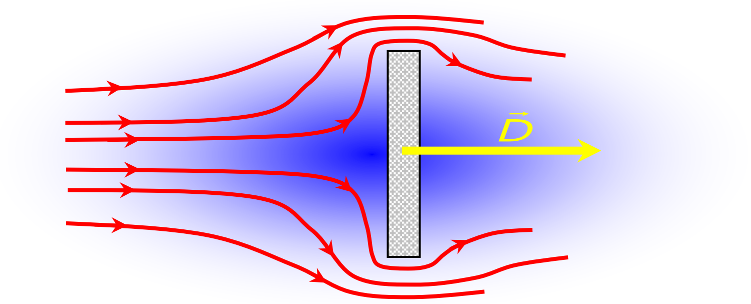
h: plate height

L: plate width

v: mean flow velocity

μ: fluid viscosity

ρ: fluid density



1. The variables F,h,L,V,μ,ρ $\Rightarrow$p = 6

M,L,T $\Rightarrow$ q=3

2. Equations in dimensions:

$$\left.\begin{aligned}
[F] &= ML\,T^{-2}\\
[\mu] &= ML^{-1}T^{-1}\\
[V] &= LT^{-1}\\
[h] &= L\\
[L] &= L\\
[\rho] &= ML^{-3}
\end{aligned}\right\}$$

3. Number of products π dimensionless: **(p- q) = 6 - 3 = 3**

4. Choice of **q= 3** dimensionally independent primary variables:

(For example h, ρ and V)

5. Formation of the **3 π** products: by combining the primary and non-primary variables.

$$\pi_1 = F\, h^{a_1} \rho^{b_1} v^{c_1} \qquad \pi_2 = L\, h^{a_2} \rho^{b_2} v^{c_2} \qquad \pi_3 = \mu\, h^{a_3} \rho^{b_3} v^{c_3}$$

6. Formulate the relationship between the 3 products **p** found:

With: $\pi_3 = \frac{\mu}{\rho v h}$ $\quad \pi_1 = \frac{F}{\rho v^2 h^2}$, $\quad \pi_2 = \frac{L}{h}$

Either :

$$F = \rho v^2 h^2\ \Phi(L/h, 1/Re)$$

Form factor

Nature of flow

III.5.1 Illustration of the benefits of the method:

If F_1 is the drag force measured on a plate of dimensions L_1 x h_1 when subjected to a flow of velocity v_1, then:

$$\frac{F_1}{\rho v_1^2 h_1^2} = \Phi(L_1/h_1, 1/Re_1) \text{ where } Re_1 = \frac{\rho v_1 h_1}{\mu}$$

Dimensional analysis using Buckingham's theorem shows that for a plate with dimensions L_2 x h_2 such that:

$L_2/h_2 = L_1/h_1$ **Similarity of form**

If $v_2 = \frac{h_1}{h_2} v_1 \Leftrightarrow v_1 h_1 = v_2 h_2 \Leftrightarrow Re_1 = Re_2$ **Hydrodynamic similarity**

Scale factor

And then $\Phi(L_1/h_1, 1/Re_1) = \Phi(L_2/h_2, 1/Re_2)$

$$\Rightarrow \frac{F_1}{\rho v_1^2 h_1^2} = \frac{F_2}{\rho v_2^2 h_2^2} \Rightarrow F_2 = \frac{v_2^2 h_2^2}{v_1^2 h_1^2} F_1 \Rightarrow \quad F_2 = F_1$$

III.6 Usual dimensionless coefficients

There are a number of dimensionless quantities which can characterise the nature of a flow:

- **The Reynolds number** $Re = \frac{\rho VL}{\mu} \quad \frac{inertia \quad forces}{viscosity \quad forces}$ (III.4)

  General importance for all types of flow

- **Tne Froude number** $Fr = \frac{V}{\sqrt{gL}} \; \frac{inertia \;\; forces}{garvity \;\; forces}$ (III.5)

  Importance for free surface flows.

- **The Euler number** $Eu = \frac{\Delta p}{\rho V^2} \; \frac{pression \;\; forces}{inertia \;\; forces}$ (III.6)

  Important if there are large pressure differences within the flow

- **The Mach number** $Ma = \frac{V}{c} \; \frac{inertia \;\; forces}{compressibility \;\; forces}$ (III.7)

  Importance for compressible fluid flows, c is the speed of sound

- **The Strouhal number** $St = \frac{\omega L}{V} \; \frac{\text{locals inertia forces}}{\text{convective inertia forces}}$ (III.8)

  Importance for non-stationary flows

III.7. Similitude in differential equations

To carry out a complete analysis of a flow, it is first necessary to make the appropriate simplifying assumptions. Evaluating the various dimensionless coefficients relating to the flow (Reynolds, Froude, etc.) will simplify the equations to be solved.

Consider the conservation of momentum equations (the Navier-Stockes equations) for an incompressible flow along x, y and z, written in the form:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \rho g \qquad \text{(III.9)}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\rho g \qquad \text{(III.10)}$$

➢ $$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g \qquad \text{(III.11)}$$

Let's consider the (z) equation (III.11) component of these equations and write it in dimensionless form:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g$$

Let's introduce some dimensionless variables:

$$\begin{cases} u^* = u/V \\ v^* = v/V \\ w^* = w/V \end{cases} \begin{cases} x^* = x/L \\ y^* = y/L \\ z^* = z/L \end{cases} p^* = p/p_0 \text{ and } t^* = t/\tau$$

Where L, V, p_0, t are characteristic quantities of the system under study.

We have :

$$\begin{cases} \frac{\partial}{\partial x}=\frac{1}{L}\frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y}=\frac{1}{L}\frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial z}=\frac{1}{L}\frac{\partial}{\partial z^*} \end{cases} \begin{cases} \frac{\partial^2}{\partial x^2}=\frac{1}{L^2}\frac{\partial^2}{\partial x^{*2}} \\ \frac{\partial^2}{\partial y^2}=\frac{1}{L^2}\frac{\partial^2}{\partial y^{*2}} \\ \frac{\partial^2}{\partial z^2}=\frac{1}{L^2}\frac{\partial^2}{\partial z^{*2}} \end{cases} \frac{\partial}{\partial t}=\frac{1}{\tau}\frac{\partial}{\partial t^*}$$

By replacing in our equation we obtain:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g$$

$$\frac{\rho V}{\tau}\frac{\partial w^*}{\partial t^*}+\frac{\rho V^2}{L}\left(u^*\frac{\partial w^*}{\partial x^*}+v^*\frac{\partial w^*}{\partial y^*}+w^*\frac{\partial w^*}{\partial z^*}\right)$$
$$=-\frac{p_0}{L}\frac{\partial p^*}{\partial z^*}+\frac{\mu V}{L^2}\left(\frac{\partial^2 w^*}{\partial x^{*2}}+\frac{\partial^2 w^*}{\partial y^{*2}}+\frac{\partial^2 w^*}{\partial z^{*2}}\right)-\rho g \qquad \text{(III.12)}$$

$$\frac{\rho V}{\tau}\frac{\partial w^*}{\partial t^*}+\frac{\rho V^2}{L}\left(u^*\frac{\partial w^*}{\partial x^*}+v^*\frac{\partial w^*}{\partial y^*}+w^*\frac{\partial w^*}{\partial z^*}\right)$$
$$=-\frac{p_0}{L}\frac{\partial p^*}{\partial z^*}+\frac{\mu V}{L^2}\left(\frac{\partial^2 w^*}{\partial x^{*2}}+\frac{\partial^2 w^*}{\partial y^{*2}}+\frac{\partial^2 w^*}{\partial z^{*2}}\right)-\rho g \qquad \text{(III.13)}$$

Divide the whole expression by: $\frac{\rho V^2}{L}$

$$\frac{L}{V\tau}\frac{\partial w^*}{\partial t^*}+u^*\frac{\partial w^*}{\partial x^*}+v^*\frac{\partial w^*}{\partial y^*}+w^*\frac{\partial w^*}{\partial z^*}$$
$$=-\frac{p_0}{\rho V^2}\frac{\partial p^*}{\partial z^*}+\frac{\mu}{\rho VL}\left(\frac{\partial^2 w^*}{\partial x^{*2}}+\frac{\partial^2 w^*}{\partial y^{*2}}+\frac{\partial^2 w^*}{\partial z^{*2}}\right)-\frac{gL}{V^2} \qquad \text{(III.14)}$$

$$St=\frac{\omega L}{V}=\frac{L}{V\tau} \quad Eu=\frac{\Delta p}{\rho V^2}=\frac{p_0}{\rho V^2} \quad Re=\frac{\rho VL}{\mu} \quad Fr=\frac{V}{\sqrt{gL}}$$

We can then write:

$$St\frac{\partial w^*}{\partial t^*}+u^*\frac{\partial w^*}{\partial x^*}+v^*\frac{\partial w^*}{\partial y^*}+w^*\frac{\partial w^*}{\partial z^*}$$
$$=-Eu\frac{\partial p^*}{\partial z^*}+\frac{1}{Re}\left(\frac{\partial^2 w^*}{\partial x^{*2}}+\frac{\partial^2 w^*}{\partial y^{*2}}+\frac{\partial^2 w^*}{\partial z^{*2}}\right)-\frac{1}{Fr^2} \qquad \text{(III.15)}$$

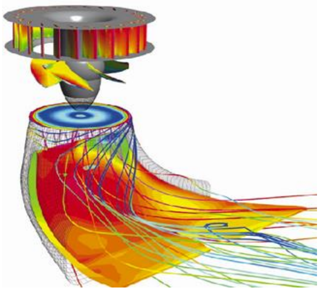
This can be interpreted as follows:

- If **St** is very small: the instantaneous derivative can be neglected and the flow can be considered stationary.
- If **Eu** is very small: the pressure gradient can be neglected.
- If **Re** is very large: the viscosity of the fluid can be neglected and it can be treated as a perfect fluid.
- If **Fr** is very large: the effects of gravity can be neglected.

III.8. Similarity and model tests

To find out about the performance of mechanical or hydraulic structures or machines (pumps, turbines, ...) before they are built or manufactured, the study is carried out on a scale model, which is a representation on a different scale of the system or structure (prototype) that you wish to test.

- The model with small-scale reproducing the actual structure
- The prototype is the structure or machine



Virtual model

Laboratory model

Prototype

The study of fluid mechanics and hydraulics problems leads to:

- **Geometric similarity**
- **Kinematic similarity**
- **Dynamic similarity**

III.8.1 Geometric similarity

For geometric similarity to exist between a model and a prototype, the length ratios must be the same and the angles between the dimensions must also be the same.

L_m: the length of the model

H_m: the height of the model

D_m: diameter of the model

A_m: the area of the model

v_m: the volume of the model

And let L_p,H_p ,D_p ,A_p and v_p be the corresponding values of the prototype.

For a geometric similarity we have:

L_r is called the scaling factor

$\frac{A_m}{A_p} = A_r$, A_r is the ratio of the areas

$\frac{v_m}{v_p} = v_r$, v_r is the ratio of the volumes

III.8.2 Kinematic similarity

Kinematic similarity is the similarity of motion.

If, at the points corresponding to the model and the prototype, the velocity and acceleration ratios are the same, as well as the velocity in the same directions, the two flows are said to be kinematically similar.

$(V_1)_m$: the fluid velocity at point 1 of the model

$(V_2)_m$: the fluid velocity at point 2 of the model

$(a_1)_m$: the fluid acceleration at point 1 of the model

$(a_2)_m$: the fluid acceleration at point 2 of the model

and $(V_1)_p$,$(V_2)_p$,$(a_1)_p$,$(a_2)_p$, the corresponding speeds and accelerations at the points in the prototype fluid.

For kinematic similarity we have :

$$\frac{(V_1)_m}{(V_1)_p} = \frac{(V_2)_m}{(V_2)_p} = V_r \qquad \text{(III.16)}$$

V_r is the velocities ratio

$$\frac{(a_1)_m}{(a_1)_p} = \frac{(a_2)_m}{(a_2)_p} = a_r \qquad \text{(III.17)}$$

a_r is the accelerations ratio

The direction of the velocity in the model and in the prototype must be the same.

III.8.3 Dynamic similarity

Dynamic similarity is the similarity of forces. The forces in the model and the prototype are similar.

If at the corresponding points the identical types of force are parallel and give the same ratio.

(Fi)m: the force of inertia at the model point

(Fv)m: the viscous force at the model point

(Fg)mthe for: the force of gravity at the model point

And (Fi)p, (Fv)p, (Fg)p are the forces corresponding to the prototype.

For dynamic similarity we have:

$$\frac{(F_i)_m}{(F_i)_p} = \frac{(F_v)_m}{(F_v)_p} = \frac{(F_g)_m}{(F_g)_p} = F_r \qquad \text{(III.18)}$$

F_r is the forces ratio

The directions of the forces in the model and in the prototype must be the same.

To ensure dynamic similarity between the model and the prototype, the dimensionless numbers of the model and the prototype must be the same.

This condition cannot be satisfied for all dimensionless numbers, so the models are designated on the basis of the forces that dominate them. This flow situation is called the law of similarity.

- **Reynolds Model Law**

In a flow situation where, in addition to inertial forces, viscous forces predominate. The similarity in the flow of the model and the prototype can be established, if the Reynolds numbers are the same for both systems.

$$\mathbf{(Re)_{model}=(Re)_{prototype}} \qquad \text{(III.19)}$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \Rightarrow \frac{\rho_m}{\rho_p} \cdot \frac{V_m}{v_p} \cdot \frac{L_m}{L_p} \frac{1}{\frac{\mu_m}{\mu_p}} = 1$$

We have $\frac{\rho_r V_r L_r}{\mu_r} = 1$ (III.20)

And $\left[\rho_r = \frac{\rho_p}{\rho_m}, V_r = \frac{V_p}{V_m}, L_r = \frac{L_p}{L_m} \right]$

Where the different index quantities r represents the scaling ratios.
In the same way:

The time scale $T_r = \frac{L_r}{V_r}$

The acceleration scale $a_r = \frac{V_r}{T_r}$

The force scale F_r= (mass X acceleration)= $m_r.a_r = \rho_r A_r V_r.a_r = \rho_r L_r^2 V^2_r$. (III.21)

The flow rate scale $q_r = (\rho AV)_r = \rho_r A_r V_r = \rho_r L_r^2 V_r$ (III.22)

Dimensional formulae of some derived quantities

Physical quantity	Expression	Dimensional formula
Area	length × breadth	$[L^2]$
Density	mass / volume	$[ML^{-3}]$
Acceleration	velocity / time	$[LT^{-2}]$
Momentum	mass × velocity	$[MLT^{-1}]$
Force	mass × acceleration	$[MLT^{-2}]$
Work	force × distance	$[ML^2T^{-2}]$
Power	work / time	$[ML^2T^{-3}]$
Energy	work	$[ML^2T^{-2}]$
Impulse	force × time	$[MLT^{-1}]$
Radius of gyration	distance	$[L]$
Pressure	force / area	$[ML^{-1}T^{-2}]$
Surface tension	force / length	$[MT^{-2}]$
Frequency	1 / time period	$[T^{-1}]$
Tension	force	$[MLT^{-2}]$
Moment of force (or torque)	force × distance	$[ML^2T^{-2}]$
Angular velocity	angular displacement / time	$[T^{-1}]$
Stress	force / area	$[ML^{-1}T^{-2}]$
Heat	energy	$[ML^2T^{-2}]$
Heat capacity	heat energy/ temperature	$[ML^2T^{-2}K^{-1}]$
Charge	current × time	$[AT]$

III.9 Applications

Exercise1:

Determine the coefficients A and B that appear in the homogeneous dimensional

equation. $\frac{d^2x}{dt^2}+A\frac{dx}{dt}+Bx=0$

with x a length and t time

Exercise2:

The pressure difference ΔP in a tube of diameter D and length l depends on the flow velocity V, the viscosity cinématique υ, the density ρ and the roughness ε of the tube surface. Using Buckingham's theorem to write the expression for ΔP in dimensionless form.

Exercise3:

Oil with a density of 920kg/m^3 and a kinematic viscosity of υ=0.003Ns/m^2 was discharged at a rate of 2500 l/s through a 1.2mm diameter tube. The tests were carried out in a 12cm diameter tube at 20°C. If the viscosity of water at 20°C is 0.01Ns/m^2, find :

1/ the flow velocity

2/ the flow rate in the model

Exercise4:

A geometrically similar model of an air duct is built at 1:25 scale and tested with water, which is 50 times more viscous and 800 times denser than air. When tested under conditions of dynamic similarity, the pressure drop in the model is 2 bar. Find the corresponding pressure drop in the prototype.

Exercise 5:

The equations of motion of a stationary flow on a flat plate are:

The equation of continuity : $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

The equation of momentum: $\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}$

If the reference quantities are u_0, ρ_0,μ_0,L_0. Write these equations in dimensionless form and determine the characteristic parameters

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