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One-Dimensional BSDEs with Jumps and Logarithmic Growth

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Abstract: In this study, we explore backward stochastic differential equations driven by a Poisson process and an independent Brownian motion, denoted for short as BSDEJs. The generator exhibits logarithmic growth in both the state variable and the Brownian component while maintaining Lipschitz continuity with respect to the jump component. Our study rigorously establishes the existence and uniqueness of solutions within suitable functional spaces. Additionally, we relax the Lipschitz condition on the Poisson component, permitting the generator to exhibit logarithmic growth with respect to all variables. Taking a step further, we employ an exponential transformation to establish an equivalence between a solution of a BSDEJ exhibiting quadratic growth in the z-variable and a BSDEJ showing a logarithmic growth with respect to y and z.

Keywords: backward stochastic differential equations; logarithmic growth; Poisson random measure; Brownian motion

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1. Introduction and Notations

Pardoux and Peng [1] initially introduced the concept of backward stochastic differential equations without the jump component, denoted briefly as BSDEs. They established the existence and uniqueness of BSDEs, assuming the Lipschitz continuity condition on the BSDE's generator with respect to both (y, z). Additionally, they assumed that the terminal value is square integrable. This result gained widespread recognition across various fields, including mathematical finance [2], finance and insurance [3], insurance reserve [4], and optimal control theory [5], as well as stochastic differential games and stochastic control [6–8]. These findings are strongly connected to partial differential equations (PDEs) [9–11]. In contrast, the latter contributions were the first to demonstrate BSDEs with random terminal time.

Given the diverse applications of BSDEs, researchers have actively worked to relax assumptions on the generator f and/or the final condition. Notably, scholars have established limited results for high-dimensional BSDEs with local Lipschitz assumptions on the driver, as demonstrated in [12–16]. While real-valued BSDEs have undergone extensive study, researchers have predominantly relied on a comparison theorem, focusing on cases where the generator grows at most linearly with respect to y and grows either linearly or quadratically in z. This has enabled the establishment of solutions under conditions of square integrability (or even integrability) for the terminal datum, as illustrated in [17–19].

In situations where the generator exhibits a quadratic growth in z (referred to as QBSDE), the existence of solutions hinges upon either boundedness or, minimally, exponential integrability of the terminal value. Various works, such as [20–22], demonstrate this requirement. Recent advancements, highlighted in [23–25], have identified a substantial class of QBSDEs for which solutions exist solely under the condition of a square-integrable terminal datum.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ represents the σ-algebra generated by two fundamental processes: a real-valued Wiener process W_t and a real-valued Poisson random measure $N(ds, de)$ defined on $[0, T] \times \Gamma$, with $\Gamma = \mathbb{R}^*$. Furthermore, we introduce $\widetilde{N}(ds, de)$ as the compensated Poisson random measure, defined by $\widetilde{N}(ds, de) = N(ds, de) - \nu(de)ds$, where ν is a σ-finite measure on Γ, equipped with its Borel field $\mathcal{B}(\Gamma)$. It is noteworthy that $\widetilde{N}$ serves as a martingale with a zero mean, referred to as the compensated Poisson random measure.

We now direct our attention to the central focus of this research endeavour. Specifically, we investigate solutions denoted as $(Y, Z, U) := (Y_t, Z_t, U_t(e))_{0\le t\le T, e\in\Gamma}$ for a BSDEJ(ζ, f). The following dynamics govern the evolution of these solutions:

$$Y_t = \zeta + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_\Gamma U_s(e)\widetilde{N}(ds, de) \tag{1}$$

The investigation initiated by Tang and Li [26] marked a pioneering achievement in the study of BSDEJ of type (1). This work demonstrated the existence and uniqueness of solutions for such equations subject to Lipschitz conditions. In a closely related context, [27] studied a class of real-valued BSDEs featuring Poisson jumps and random time horizons. They proved the existence of at least one solution for BSDEs characterized by a driver exhibiting linear growth.

In subsequent work, ref. [28] extended these discoveries by proving the existence but not the uniqueness of solutions for BSDEs with jumps. They considered continuous coefficients that satisfy an extended linear growth condition in this extension. This result was generalized to situations where the generators are either left- or right-continuous.

Recent advancements in research by [29,30] strengthen the connections between specific classes of quadratic BSDEs and conventional BSDEs driven by continuous functions. Notably, ref. [31] made an important contribution by proving the well-posedness of solutions under local Lipschitz conditions, with special emphasis on the Brownian motion component. They also demonstrated the existence of one and only one solution for a class of nonlinear variants of the backward Kolmogorov equation.

Previous studies formulated all of the above results for one-dimensional BSDEs. Ref. [32] studied a multidimensional Markovian BSDEJ and demonstrated that a given Poisson process and deterministic functions can express the adapted solution. They established the existence of solutions for these equations under the assumption that their generators are either continuous with respect to y and z and Lipschitz in u or continuous in all their variables and adhere to standard linear growth assumptions. Bahlali and El Asri [33] investigated situations where the generator of the BSDEs is bounded by $(|z|\sqrt{|\ln|z||})$. They also considered the terminal value, assuming it to be merely $\mathbb{L}^p$-integrable, with $p > 2$. However, the extension of this condition was recently explored by [34], who supposed that the drift is dominated by $(|y||\ln|y|| + |z|\sqrt{|\ln|z||})$. Additionally, refs. [31,35] studied BSDEs associated with jump Markov processes, with the latter work presenting a proof under assumptions different from those considered in the present study.

Logarithmic growth generators in BSDEs play a crucial role in financial risk management, capturing the common assumption of asset growth proportional to their current value. Studying BSDEs with such generators is essential for optimizing investment portfolios, pricing, hedging derivatives, modeling energy prices, and guiding optimal investment strategies in wealth accumulation problems (see [33,36]). Moreover, their connection to partial differential equations (PDEs) with logarithmic coefficients (as explored in [34]) highlights their relevance in physics. Notably, the logarithmic growth condition's weaker nature compared to quadratic and super-linear ones further emphasizes its versatility. Consequently, understanding and solving one-dimensional BSDEs with logarithmic growth generators are fundamental for tackling complex problems across finance, engineering, physics, and even biology.

In this work, we proceed according to the following methodology. We establish the existence and uniqueness of the solution for BSDEJs whose generators show a growth described by a logarithmic function of the type $(|y||\ln|y||+|z|\sqrt{|\ln|z||})$ but keeping the linear growth condition in u. Initially, we present a priori estimates for solutions of BSDEs, followed by the presentation of the main result; this comprises the content of Section 2. Section 3 extends the logarithmic growth condition for BSDEJs, specifically by relaxing the Lipschitz condition on the jump coefficient. In Section 4, we demonstrate the equivalence of previously obtained solutions through an exponential transformation. Finally, Section 5 provides the conclusion of our work. Some detailed proofs of crucial lemmas are gathered in Appendix A.

Below, we list some notations that will be used in this paper.

For a specified $T \geq 0$, the following notation is employed:

- $\mathcal{P}$: represents the predictable σ-field on $[0,T]\times\Omega$.
- $\widetilde{\Omega}$: is defined as $[0,T]\times\Omega\times\Gamma$.
- $\mathcal{E} := \mathcal{B}(\Gamma)$.
- $\widetilde{\mathcal{P}} := \mathcal{P}\otimes\mathcal{E}$ denotes the predictable σ-algebra on $\widetilde{\Omega}$.

In the subsequent sections of this work, we shall introduce useful functional spaces: For $m \geq 1$:

- $\mathcal{S}^m([s,t];\mathbb{R})$: the space of $\mathbb{R}$-valued adapted càdlàg processes Y such that

$$|Y|^m_{\mathcal{S}} = \mathbb{E}\big[\sup_{s\leq r\leq t}|Y_r|^m\big] \;<\; \infty.$$

- $\mathcal{S}^\infty([s,t];\mathbb{R})$: the space of $\mathbb{R}$-valued adapted càdlàg processes Y such that

$$|Y|_{\mathcal{S}^\infty} = \operatorname*{ess\,sup}_{s\leq r\leq t}|Y_r| < \; \infty.$$

- $\mathbb{H}^m([s,t];\mathbb{R})$: the space of $\mathbb{R}$-valued predictable processes satisfying

$$\int_s^t \mathbb{E}\big[|Z_r|^m\big]dr \;<\; \infty.$$

- $L^2(\Gamma,\mathcal{E},\nu;\mathbb{R})$: the space of Borelian functions $\ell:\Gamma\to\mathbb{R}$ such that

$$\|\ell\|_\nu = \Big(\int_\Gamma |\ell(e)|^2\nu(de)\Big)^{1/2} \;<\; \infty.$$

- $\mathbb{L}^m([s,t],\nu;\mathbb{R})$: the set of the processes $U:\widetilde{\Omega}\to\mathbb{R}$ is $\widetilde{\mathcal{P}}$ -measurable and

$$\int_s^t \mathbb{E}\big[\|U_r\|^m_\nu\big]dr \;<\; \infty.$$

2. Existence and Uniqueness of Solutions

In this section, we establish the foundational assumption that forms the basis of our analysis, providing a framework for subsequent developments. This assumption is pivotal for exploring solutions to the BSDEJ Equation (1). We then introduce preliminary estimates of the solution and delineate key lemmas crucial for establishing both the existence and uniqueness of solutions.

Assumption 1.

(A.1) *Assume that $\mathbb{E}[|\zeta|^{\mu_T+1}]$ is finite, where $\mu_t := e^{\theta t}$ for all $t\in[0,T]$ and θ is a sufficiently large positive constant.*

(A.2) *(i) f is continuous in (y,z) and Lipschitz with respect to u (t,ω)-a.e.*

(ii) *There exist constants c_0, c_1, c_2, C_{Lip}, and a positive process ϑ such that*

$$\int_0^T \mathbb{E}\big[\vartheta_s^{\mu_s+1}\big]ds \quad < \quad +\infty.$$

Additionally, for every $t, \omega, y, z, u, u_1, u_2$:

$$|\ f(t,\omega,y,z,u)\ | \quad \leq \quad \vartheta_t + g_{1,c_2}(y) + g_{2,c_0}(z) + c_1\|u\|_\nu,$$

and

$$|f(t,\omega,y,z,u_1) - f(t,\omega,y,z,u_2)| \quad \leq \quad C_{Lip}\|u_1 - u_2\|_\nu,$$

where $g_{1,c_2}(y) = c_2|y||\ln|y||$ and $g_{2,c_0}(z) = c_0|z|\sqrt{|\ln|z||}$.

(A.3) *There exists a sequence of real numbers $(A_N)_{N>1}$ along with constants $M_2 \in \mathbb{R}_+$, $r > 0$, satisfying:*

(i) *For every integer $N > 1$, we have $1 < A_N \leq N^r$.*
(ii) $\lim_{N\to\infty} A_N = \infty$.
(iii) *For any natural number $N \in \mathbb{N}$, and every $y_1,\ y_2,\ z_1,\ z_2,\ u$ such that: $|y_1|,\ |y_2|,\ |z_1|,\ |z_2|,\ \|u\|_\nu \leq N$, the following holds:*

$$\begin{aligned}&(y_1 - y_2)\big(f(t,\omega,y_1,z_1,u) - f(t,\omega,y_2,z_2,u)\big)\\ \leq\quad & M_2\left(\ |\ y_1 - y_2\ |^2 \ln(A_N) +\ |\ y_1 - y_2\ |\ |z_1 - z_2|\sqrt{\ln(A_N)} + \frac{\ln(A_N)}{A_N}\ \right).\end{aligned}$$

Definition 1. *A solution to the BSDEJ(ζ, f) is a triplet*

$$(Y,Z,U) \in \mathcal{S}^{\mu_T+1}([0,T];\mathbb{R}) \times \mathbb{H}^2([0,T];\mathbb{R}) \times \mathbb{L}^2([0,T],\nu;\mathbb{R})$$

that satisfies Equation (1).

2.1. Technical Lemmas

This subsection introduces four technical lemmas needed in the sequel. More precisely, the first three are crucial in proving the results of the next subsection. Their proofs are provided in Appendix A.

Lemma 1. *Let y, $z \in \mathbb{R}$ such that $|y| > e$. For any positive constant C_1, there exists another positive constant C_2 such that the following inequality holds:*

$$C_1|y||z|\sqrt{|\ln|z||} \leq \frac{|z|^2}{2} + C_2|y|^2\ln|y|. \tag{2}$$

Lemma 2. *For $p \in (0,\infty)$ and $x, y \in \mathbb{R}$, the following inequality holds:*

$$\int_0^1 (1-a)|x+ay|^p da \geq 3^{-(1+p)}|x|^p.$$

Lemma 3. *Let (Y,Z,U) be a solution to the BSDEJ (1). Under (**A.1**) and (**A.2**), there exists a positive constant C such that*

$$\begin{aligned}&\mathbb{E}\Big[|Y_t|^{\mu_t+1} + \int_t^T \mu_s(\mu_s+1)|Y_s|^{\mu_s-1}(|Z_s|^2 + ||U_s||_\nu^2)ds\Big]\\ &\leq C\Big(1 + \mathbb{E}[|\zeta|^{\mu_T+1}] + (\mu_T+1)^{\mu_T}\int_0^T \mathbb{E}[\vartheta_s^{\mu_s+1}]ds\Big).\end{aligned}$$

Lemma 4. *Let (**A.1**), (**A.2**)-(ii) be satisfied. Then, there exists a positive constant* $C(T, \alpha, c_0, c_1, c_2)$ *such that*

$$\int_0^T \mathbb{E}\big[|f(s, Y_s, Z_s, U_s)|^{\frac{2}{\alpha}}\big]ds \quad \leq \quad \widetilde{K}_1,$$

where $1 < \alpha < 2$, *and*

$$\widetilde{K}_1 := C(T, \alpha, c_0, c_1, c_2)\bigg(1 + \int_0^T \mathbb{E}\big[\vartheta_s^2 + |Y_s|^{\mu_s+1} + |Z_s|^2 + \|U_s\|_\nu^2\big]ds\bigg).$$

2.2. A Priori Estimates

This subsection aims to give some prior estimates for the solutions of BSDEJ (1). These estimates establish bounds on the solutions, ensuring that if the solutions exist, they will belong to some appropriate spaces.

Lemma 5. *Consider a solution* (Y, Z, U) *to the BSDEJ (1). Additionally, assume that the pair* (ζ, f) *satisfies conditions (**A.1**) and (**A.2**). In this context, we establish the existence of a universal constant* $C(T, c_0, c_1, c_2)$*, as follows:*

(i) $\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^{\mu_t+1}\Big] \leq \widetilde{K}_2.$

(ii) $\int_0^T \mathbb{E}\big[|Z_s|^2 + \|U_s\|_\nu^2\big]ds \leq \widetilde{K}_3,$

where

$$\widetilde{K}_2 := C(T, c_0, c_1, c_2)\Big(1 + \mathbb{E}[|\zeta|^{\mu_T+1}] + \int_0^T \mathbb{E}[\vartheta_s^{\mu_s+1}]ds\Big),$$
$$\widetilde{K}_3 := C(T, c_0, c_1, c_2)\Big(1 + T\widetilde{K}_2 + \mathbb{E}\big[|\zeta|^2\big] + \int_0^T \mathbb{E}\big[\vartheta_s^2\big]ds\Big).$$

The first lemma that follows allows for a localization procedure introduced to establish solutions' existence and uniqueness. The second one provides a prior estimate for the approximating solutions and guarantees that these solutions do not diverge. The proofs for these lemmas can be performed and adapted to our setting similarly as outlined in [34].

Lemma 6. *There exists* (f_n)*, a sequence of functions, satisfying:*

(i) For every n*, the functions* f_n *are bounded and exhibit global Lipschitz continuity with respect to* (y, z, u) *for a.e.* t *and* $\mathbb{P}$ *-a.s.*

(ii) $\sup_n |f_n(t, \omega, y, z, u)| \leq \vartheta_t + g_{1,c_2}(y) + g_{2,c_0}(z) + c_1\|u\|_\nu.$

(iii) For each N, $\rho_N(f_n - f) \longrightarrow 0$ *as* $n \longrightarrow \infty$*, where*

$$\rho_N(f) \quad = \quad \mathbb{E}\Big[\int_0^T \sup_{|y|,|z|,\|u\|_\nu \leq N} |f(s, y, z, u)|ds\Big].$$

Lemma 7. *Consider* f *and* ζ *as defined in Lemma 5. Let* (f_n) *denote the sequence of functions associated with* f *by Lemma 6. Let* (Y^n, Z^n, U^n) *represent the solution to the BSDEJ*(ζ, f_n)*. Consequently, we have :*

(a) $\sup_n \mathbb{E}[\int_0^T \|U_s^n\|_\nu^2 ds] \leq K_1.$

(b) $\sup_n \mathbb{E}[\sup_{0\leq t\leq T} |Y_t^n|^{\mu_T+1}] \leq K_2.$

(c) $\sup_n \mathbb{E}[\int_0^T |Z_s^n|^2 ds] \leq K_3.$

(d) $\sup_n \mathbb{E}[\int_0^T |f_n(s, Y_s^n, Z_s^n, U_s^n)|^{\frac{2}{\alpha}} ds] \leq K_4.$

where K_1*,* K_2*,* K_3*, and* K_4 *are constants independent of* n*.*

2.3. Some Convergence Results

This subsection establishes estimates between two potential solutions. This analysis is essential for demonstrating the existence of solutions and understanding the properties of these solutions in the context of the study on one-dimensional BSDEs with logarithmic growth. Moving forward, we use the notation $\hat{h}_s^{n,m}$ to represent the difference between h_s^n and h_s^m for any given quantities.

Proposition 1. *For every $R \in \mathbb{N}$, $\beta \in (1, 3-\alpha)$, $0 < \delta < \frac{\beta-1}{2M_2^2 + C_{Lip}^2} \min(\frac{1}{2}, \frac{\kappa}{r\beta})$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $S \leq T$:*

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big[\sup_{(S-\delta)^+ \leq t \leq S} |\hat{Y}_t^{n,m}|^\beta + \int_{(S-\delta)^+}^S \frac{(|\hat{Z}_s^{n,m}|^2 + \|\hat{U}_s^{n,m}\|_\nu^2)}{(|\hat{Y}_s^{n,m}|^2 + \Lambda_R)^{\frac{2-\beta}{2}}} ds\Big] \leq \varepsilon + \frac{\ell}{\beta-1} e^{C_N \delta} \limsup_{n,m\to+\infty} \mathbb{E}\big[|\hat{Y}_S^{n,m}|^\beta\big].$$

Here , $\Lambda_R = \sup\{(A_N)^{-1}, N \geq R\}$, $C_N := \frac{\beta}{\beta-1}(2M_2^2 + C_{Lip}^2)\ln(A_N)$, and ℓ is a positive constant. The definition of κ can be found below.

We rely on Lemma A1 to substantiate the preceding proposition.

Proof. We define the constant C in Lemma A1 as $C_N := C_{N,1} + C_{N,2}$, where $C_{N,1} := \frac{2M_2^2\beta}{\beta-1}\ln(A_N)$ and $C_{N,2} := \frac{C_{Lip}^2\beta}{\beta-1}\ln(A_N)$. Additionally, let $\gamma := \delta C_N (\ln(A_N))^{-1}$. We will examine the following quantity:

$$\begin{aligned}
&-C_N \int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}} ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds - \beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds \\
&+J_{3,t} + J_{4,t}.
\end{aligned}$$

The control of the expression involving the process $(\hat{Z}_s^{n,m})$ has been postponed to Lemma A2, where Young's inequality plays a crucial role, leading us to the following:

$$\begin{aligned}
&-C_{N,1}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}} ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds + J_{3,t} \\
\leq\ & -\beta\frac{(\beta-1)}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds.
\end{aligned}$$

We direct our attention to the expression encompassing the norm $\|\hat{U}_s^{n,m}\|_\nu$.

By applying Young's inequality and setting $C_{N,2} = \beta\frac{C_{Lip}^2}{\beta-1}\ln(A_N)$ for sufficiently large A_N (i.e., $A_N \geq e$), we obtain the following result:

$$\begin{aligned}
&-C_{N,2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}} ds + \beta C_{Lip}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}|\,\|\hat{U}_s^{n,m}\|_\nu ds \\
&-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds \\
\leq\ & -\beta\frac{\beta-1}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds. \qquad (3)
\end{aligned}$$

Based on Lemma 3 and employing Burkholder–Davis–Gundy's inequality and Hölder's inequality, while taking into account the relationship $\frac{\beta-1}{2} + \frac{\kappa}{2} + \frac{\alpha}{2} = 1$ as well as the inequalities (A7) and (3), we obtain a positive universal constant ℓ such that, for all $\delta > 0$, the following inequality universally holds:

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(S-\delta)^+\leq t\leq S}\big[e^{C_N t}\varphi_t^{\frac{\beta}{2}}\big]\Big]+\mathbb{E}\Big[\int_{(S-\delta)^+}^{S} e^{C_N s}\varphi_s^{\frac{\beta}{2}-1}(|\hat{Z}_s^{n,m}|^2+\|\hat{U}_s^{n,m}\|_\nu^2)ds\Big] \\
\leq\;& \frac{\ell}{\beta-1}e^{C_N\delta}\Big\{\mathbb{E}\big[\varphi_S^{\frac{\beta}{2}}\big]+\frac{\beta}{N^\kappa}\Big[\mathbb{E}\int_0^T\varphi_s ds\Big]^{\frac{\beta-1}{2}}\Big[\mathbb{E}\int_0^T\Phi^2(s)ds\Big]^{\frac{\kappa}{2}} \\
&\times\Big[\mathbb{E}\int_0^T|f_n(s,Y_s^n,Z_s^n,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^n)|^{\frac{2}{\alpha}}ds\Big]^{\frac{\alpha}{2}} \\
&+\beta[4N^2+\Lambda_1]^{\frac{\beta-1}{2}}\mathbb{E}\Big[\int_0^T\sup_{|y|,|z|,\|u\|_\nu\leq N}|f_n(s,y,z,u)-f(s,y,z,u)|ds \\
&+\int_0^T\sup_{|y|,|z|,\|u\|_\nu\leq N}|f_m(s,y,z,u)-f(s,y,z,u)|ds\Big]\Big\}.
\end{aligned}$$

Utilizing Lemmas 6 and 7, for any $N > R$:

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(S-\delta)^+\leq t\leq S}|\hat{Y}_t^{n,m}|^\beta+\mathbb{E}\int_{(S-\delta)^+}^{S}\frac{(|\hat{Z}_s^{n,m}|^2+\|\hat{U}_s^{n,m}\|_\nu^2)}{(|\hat{Y}_s^{n,m}|^2+\Lambda_R)^{\frac{2-\beta}{2}}}ds\Big] \\
\leq\;& \frac{\ell}{\beta-1}e^{C_N\delta}\mathbb{E}\big[|\hat{Y}_S^{n,m}|^\beta\big]+\frac{\ell}{\beta-1}\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}} \\
&+\frac{4\ell}{\beta-1}\beta K_4^{\frac{\alpha}{2}}(4TK_2+T\Lambda_R)^{\frac{\beta-1}{2}}(8TK_2+16K_1+16K_3)^{\frac{\kappa}{2}}\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}} \\
&+\frac{2\ell}{\beta-1}e^{C_N\delta}\beta[2N^2+\Lambda_1]^{\frac{\beta-1}{2}}[\rho_N(f_n-f)+\rho_N(f_m-f)].
\end{aligned}$$

Given $\delta < \frac{\beta-1}{2M_2^2+C_{Lip}^2}\min\left(\frac{1}{2},\frac{\kappa}{r\beta}\right)$, we can derive

$$\lim_{N\to\infty}\left(\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}}+\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}}\right) = 0.$$

To complete the proof of Proposition 1, we commence by taking the limits as n, m approach their respective limits $+\infty, +\infty$ followed by a subsequent limit as N tends to infinity, in accordance with assertion (iii) of Lemma 6. □

2.4. The Main Result

The primary focus of this work is to investigate the existence and the uniqueness results of solutions for BSDEJ (1) under Assumption 1.

Theorem 1. *Under Assumption 1, Equation (1) admits one and only one solution (Y, Z, U) in $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$.*

Proof of existence. By applying Proposition 1 successively with $S = T$, $S = (T-\delta)^+$, $S = (T-2\delta)^+\ldots$ and utilizing the Lebesgue dominated convergence theorem, we can show that for any $\beta \in (1, 3-\alpha)$, the following holds:

$$\limsup_{n,m\to+\infty}\mathbb{E}\Big[\sup_{0\leq t\leq T}|\hat{Y}_t^{n,m}|^\beta+\int_0^T\frac{(|\hat{Z}_s^{n,m}|^2+\|\hat{U}_s^{n,m}\|_\nu^2)}{(|\hat{Y}_s^{n,m}|^2+\Lambda_R)^{\frac{2-\beta}{2}}}ds\Big]=0.$$

Through the application of the Cauchy–Schwarz inequality, we derive

$$\mathbb{E}\Big[\int_0^T (|\hat{Z}_s^{n,m}| + \|\hat{U}_s^{n,m}\|_\nu)ds\Big] \quad \leq \quad \sqrt{2}\Big(\mathbb{E}\Big[\int_0^T \frac{(|\hat{Z}_s^{n,m}|^2 + \|\hat{U}_s^{n,m}\|_\nu^2)}{(|\hat{Y}_s^{n,m}|^2 + \Lambda_R)^{\frac{2-\beta}{2}}} ds\Big]\Big)^{\frac{1}{2}}$$
$$\times\Big(\mathbb{E}\Big[\int_0^T \Big(|\hat{Y}_s^{n,m}|^2 + \Lambda_R\Big)^{\frac{2-\beta}{2}} ds\Big]\Big)^{\frac{1}{2}}.$$

It is evident from Lemma 7 that

$$\Bigg(\mathbb{E}\Big[\int_0^T \Big(|\hat{Y}_s^{n,m}|^2 + \Lambda_R\Big)^{\frac{2-\beta}{2}} ds\Big]\Bigg)^{\frac{1}{2}} \; < \; \infty.$$

Consequently,

$$\lim_{n,m\to+\infty} \mathbb{E}\Big[\sup_{0\leq t\leq T} |\hat{Y}_t^{n,m}|^\beta + \int_0^T \big(|\hat{Z}_s^{n,m}| + \|\hat{U}_s^{n,m}\|_\nu\big)ds\Big] \quad = \quad 0.$$

Thus, there exists (Y, Z, U) that satisfies

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t|^\beta + \int_0^T \big(|Z_s| + \|U_s\|_\nu\big)ds\Big] \quad < \quad \infty,$$

and

$$\lim_{n\to+\infty} \mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t^n - Y_t|^\beta + \int_0^T \big(|Z_s^n - Z_s| + \|U_s^n - U_s\|_\nu\big)ds\Big] \quad = \quad 0.$$

Specifically, a sub-sequence denoted as (Y^n, Z^n, U^n) exists, such that

$$\lim_{n\to+\infty} (|Y_t^n - Y_t| + |Z_t^n - Z_t| + \|U_t^n - U_t\|_\nu) \; = \; 0 \quad a.e.\ (t,\omega). \tag{4}$$

We still need to establish the convergence in probability of the following term:

$$\int_0^T (f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s))ds,$$

as n approaches ∞. The initial step is applying the triangular inequality, which yields

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T |f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)|ds\Big] \\
\leq \;\; &\mathbb{E}\Big[\int_0^T |f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s^n, Z_s^n, U_s^n)|ds\Big] \\
&+\mathbb{E}\Big[\int_0^T |f(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)|ds\Big].
\end{aligned}$$

Utilizing Hölder's inequality and the following inequality,

$$\mathbb{1}_{\{|Y_s^n|+|Z_s^n|+\|U_s^n\|_\nu\geq N\}} \quad \leq \quad \frac{(|Y_s^n| + |Z_s^n| + \|U_s^n\|_\nu)^{2-\alpha}}{N^{2-\alpha}},$$

we obtain

$$
\begin{aligned}
& \mathbb{E}\Big[\int_0^T |(f_n - f)(s, Y_s^n, Z_s^n, U_s^n)|ds\Big] \\
\leq\ & \mathbb{E}\Big[\int_0^T |(f_n - f)(s, Y_s^n, Z_s^n, U_s^n)|\ \mathbb{1}_{\{|Y_s^n|+|Z_s^n|+\|U_s^n\|_\nu<N\}}ds\Big] \\
& +\mathbb{E}\Big[\int_0^T |(f_n - f)(s, Y_s^n, Z_s^n, U_s^n)|\frac{(|Y_s^n| + |Z_s^n| + \|U_s^n\|_\nu)^{2-\alpha}}{N^{2-\alpha}}\ \mathbb{1}_{\{|Y_s^n|+|Z_s^n|+\|U_s^n\|_\nu\geq N\}}ds\Big] \\
\leq\ & \rho_N(f_n - f) + \frac{4K_4^{\frac{\alpha}{2}}(TK_2 + K_1 + K_3)^{1-\frac{\alpha}{2}}}{N^{2-\alpha}}.
\end{aligned}
$$

The last inequality is obtained from Lemmas 6 and 7. Taking the limit successively first with respect to n and then to N in the preceding inequality, we arrive at

$$\lim_n \mathbb{E}\Big[\int_0^T |f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s^n, Z_s^n, U_s^n)|ds\Big] \quad = \quad 0.$$

Considering the limit (4) and the continuity of the function f with respect to (y, z, u) for all $t \in [0, T]$, we obtain

$$\lim_n |f(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)| \quad = \quad 0. \quad a.e.\ (t, \omega).$$

Furthermore, Lemma 4 and the conditions (a–c) outlined in Lemma 7 affirm the uniform integrability of the sequence

$$|f(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)|.$$

As a result:

$$\lim_{n\to\infty} \int_0^T \mathbb{E}|f(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)|ds \quad = \quad 0.$$

Consequently, the BSDE (1) has a solution in $\mathcal{S}^\beta([0, T]; \mathbb{R}) \times \mathbb{H}^1([0, T]; \mathbb{R}) \times \mathbb{L}^1([0, T], \nu; \mathbb{R})$. Taking account of Lemma 5, we conclude that it belongs to $\mathcal{S}^{\mu_T+1}([0, T]; \mathbb{R}) \times \mathbb{H}^2([0, T]; \mathbb{R}) \times \mathbb{L}^2([0, T], \nu; \mathbb{R})$. This achieves the proof of the existence part. □

Proof of uniqueness. Consider two solutions (Y, Z, U) and (Y', Z', U') to the BSDEJ (1). Drawing from the proof of Proposition 1, it can be demonstrated that for every $R > 2$,

$$\beta \in (1, 3 - \alpha), \ \delta \quad < \quad \frac{\beta - 1}{2M_2^2 + C_{\text{Lip}}^2} \min\Big(\frac{1}{2}, \frac{\kappa}{r\beta}\Big) \quad \text{and} \quad \varepsilon > 0,$$

there is an $N_0 > R$, for all subsequent $N > N_0$ and each $S \leq T$:

$$
\begin{aligned}
& \mathbb{E}\Big[|Y_t - Y_t'|^\beta\Big] + \mathbb{E}\Big[\int_{(S-\delta)^+}^S \Big(|Z_s - Z_s'|^2 + \|U_s - U_s'\|_\nu^2\Big)\Big(|Y_s - Y_s'|^2 + \Lambda_R\Big)^{\frac{\beta-2}{2}} ds\Big] \\
\leq\ & \varepsilon + \frac{\ell}{\beta - 1} e^{C_N\delta}\mathbb{E}\Big[|Y_S - Y_S'|^\beta\Big].
\end{aligned}
$$

We successively set $S = T$, followed by updating S as $S = (T - \delta)^+$, and so on. Thus, the BSDEJ (1) has a unique solution $(Y, Z, U) \in \mathcal{S}^{\mu_T+1}([0, T]; \mathbb{R}) \times \mathbb{H}^2([0, T]; \mathbb{R}) \times \mathbb{L}^2([0, T], \nu; \mathbb{R})$. □

Example 1. *Let* $g(t, \omega, y, z) := \vartheta_t + c_2|y||\ln|y|| + c_0|z|\sqrt{|\ln(|z|)|} + \|u\|_\nu$. *Clearly, g satisfies (**A.2**), so we will now verify that (**A.3**) holds true:*

Indeed, letting $g_{1,c_2}(y) := c_2|y||\ln|y||$; $g_{2,c_0}(z) := c_0|z|\sqrt{|\ln|z||}$, *we have*

$$\begin{aligned} g(t,\omega,y_1,z_1,u) - g(t,\omega,y_2,z_2,u) &= g_{1,c_2}(y_1) - g_{1,c_2}(y_2) \\ &\quad + g_{2,c_0}(z_1) - g_{2,c_0}(z_2). \end{aligned}$$

We shall examine the function g_{1,c_2} *under the following conditions:*

$$0 \le |y_1|, |y_2| \le \frac{1}{N} \text{ and } \frac{1}{N} \le |y_1|, |y_2| \le N.$$

Additionally, we will analyze g_{2,c_0} *across various cases:*

$$\begin{cases} 0 \le |z_1|,\ |z_2| \le \frac{1}{N}, & \frac{1}{N} \le |z_1|,\ |z_2| \le 1-\tilde{\epsilon}, \\ 1-\tilde{\epsilon} \le |z_1|,\ |z_2| \le 1+\tilde{\epsilon}, & 1+\tilde{\epsilon} \le |z_1|,\ |z_2| \le N, \end{cases}$$

where $\tilde{\epsilon} \in (0,1)$ *is small enough, and* N *is sufficiently large.*

Clearly, in the first case ($|y|,\ |z| \le \frac{1}{N}$), *the two functions satisfy* (***A.3***),

$$\begin{aligned} |g_{1,c_2}(y_1) - g_{1,c_2}(y_2) + g_{2,c_0}(z_1) - g_{2,c_0}(z_2)| &\le |g_{1,c_2}(y_1)| + |g_{1,c_2}(y_2)| + |g_{2,c_0}(z_1)| + |g_{2,c_0}(z_2)| \\ &\le \max(c_0,c_2)\frac{4}{N}\ln(N). \end{aligned}$$

The mean value theorem, applied in the second term, implies the following:

$$\begin{aligned} |g_{1,c_2}(y_1) - g_{1,c_2}(y_2) + g_{2,c_0}(z_1) - g_{2,c_0}(z_2)| &\le |g_{1,c_2}(y_1) - g_{1,c_2}(y_2)| + |g_{2,c_0}(z_1) - g_{2,c_0}(z_2)| \\ &\le \max(c_0,c_2)\Big(|y_1-y_2|\ln(N) + |z_1-z_2|\sqrt{\ln(N)}\Big). \end{aligned}$$

Applying the mean value theorem again, we can prove the remaining cases for the function g_{2,c_0}. *Therefore,* (***A.3***) *holds for* $A_N = N$.

Further examples can be found in [37].

3. Generalized Logarithmic Growth Condition for BSDEs with Jumps

Now, we examine a distinct BSDE with jumps from the one in (1), introducing different assumptions for the generator of the next BSDEJ:

$$Y_t = \zeta + \int_t^T f\big(s,Y_s,Z_s,\textstyle\int_\Gamma U_s(e)\nu(de)\big)ds - \displaystyle\int_t^T Z_s dW_s - \int_t^T \int_\Gamma U_s(e)\widetilde{N}(ds,de). \tag{5}$$

Assumption 2.

(***A.1***)′ *Assume that* $\mathbb{E}[|\zeta|^{\mu_T+1}]$ *is finite, where* $\mu_t := e^{\theta t}$ *for all* $t \in [0,T]$ *and* θ *is a sufficiently large positive constant.*

(***A.2***)′ *(i)* *For almost all* (t,ω), *the function* f *is continuous with respect to* (y,z,u).

(ii) *There exists a positive process* ϑ *such that*

$$\int_0^T \mathbb{E}\big[\vartheta_s^{\mu_s+1}\big]ds < +\infty.$$

Additionally, for every t, y, z, *and* u,

$$\big|f\big(t,y,z,\textstyle\int_\Gamma u(e)\nu(de)\big)\big| \le \vartheta_t + g_{1,c_2}(y) + g_{2,c_0}(z) + g_{3,c_1}(u),$$

where $g_{3,c_1}(u) = c_1\|u\|_\nu\sqrt{|\ln\|u\|_\nu|}$, c_0, c_1 *and* c_2 *are positive constants.*

(**A.3**)′ *There exists a real-valued sequence* $(A_N)_{N>1}$ *and constants* $M_2 \in \mathbb{R}_+$, $r > 0$ *such that*

- *(i)* *For every integer* $N > 1$, *we have* $1 < A_N \le N^r$.
- *(ii)* $\lim_{N\to\infty} A_N = \infty$.
- *(iii)* *For every* $N \in \mathbb{N}$, *and every* $y_1,\ y_2,\ z_1,\ z_2,\ u_1,\ u_2$ *such that* $|y_1|,\ |y_2|,\ |z_1|,\ |z_2|,\ \|u_1\|_\nu,\ \|u_2\|_\nu \le N$, *we have*

$$
\begin{aligned}
& (y_1 - y_2)\big(f\big(t,\omega,y_1,z_1,\textstyle\int_\Gamma u_1(e)\nu(de)\big) - f\big(t,\omega,y_2,z_2,\textstyle\int_\Gamma u_2(e)\nu(de)\big)\big) \\
\le\ & M_2\Big(\mid y_1 - y_2 \mid^2 \ln(A_N) + \mid y_1 - y_2 \mid \sqrt{\ln(A_N)}\big(\mid z_1 - z_2 \mid + \|u_1 - u_2\|_\nu\big) \\
& + \frac{\ln(A_N)}{A_N}\Big).
\end{aligned}
$$

By following the steps outlined in the previous proofs, we can obtain a unique solution for BSDEJ (5) in which the transaction with u becomes proportionally identical to the transaction with z.

The previous lemmas maintain their validity while adhering to (5) and Assumption 2. Therefore, we will provide concise proofs, building upon the earlier derivations.

The proof of Lemma 5 under Assumption 2 is provided in Appendix A.

In what follows, we state a lemma concerning the stability result for the solution of BSDEJ (5). The proof follows the same steps as Lemma 3.5 in [34].

Lemma 8. *There exists a sequence of functions* (f_n) *with the following properties:*

- *(i)* *For each* n, f_n *is bounded and globally Lipschitz in* (y,z,u) *a.e.* t *and* $\mathbb{P}$*-a.s.*ω.
- *(ii)* *Moreover, for all* n, *we have* $\mathbb{P}$*-a.s., a.e.* $t \in [0,T]$:

$$
\sup_n \big|f_n\big(t,\omega,y,z,\textstyle\int_\Gamma u(e)\nu(de)\big)\big| \quad \le \quad \vartheta_t + g_{1,c_2}(y) + g_{2,c_0}(z) + g_{3,c_1}(u).
$$

- *(iii)* *Additionally, for every* N, *as* n *tends to infinity, the quantity* $\rho_N(f_n - f)$ *converges to* 0, *where*

$$
\rho_N(f) \quad = \quad \mathbb{E}\Big[\int_0^T \sup_{|y|,|z|,\|u\|_\nu \le N} \big|f_n\big(s,\omega,y,z,\textstyle\int_\Gamma u(e)\nu(de)\big)\big|\, ds\Big].
$$

Proposition 2. *Proposition 1, which establishes the estimate between two solutions, maintains its validity within this section despite variations in the values of* δ *and* C, *as presented in the subsequent lemma.*

Lemma 9. *Assuming that* $C := C_N := 3\beta\frac{M_2^2}{\beta-1}\ln(A_N)$ *and* $\delta < \frac{\beta-1}{3M_2^2}\min(\frac{1}{2},\frac{\kappa}{r\beta})$, *for any* $S \le T$ *we have*

$$
\begin{aligned}
e^{Ct}\varphi_t^{\frac{\beta}{2}} + C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \widetilde{\mathbb{M}}_t \quad \le \quad & e^{CS}\varphi_S^{\frac{\beta}{2}} - \frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds \\
& -\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds \\
& +\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\
& +\mathbb{M}_t + \tilde{J}_{1,t} + \tilde{J}_{2,t} + \tilde{J}_{3,t},
\end{aligned}
$$

where

$$\tilde{J}_{1,t} := \beta e^{CS}\frac{1}{N^\kappa}\int_t^S \varphi_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)\Big|f_n\big(s,Y_s^n,Z_s^n,\textstyle\int_\Gamma U_s^n(e)\nu(de)\big) - f_m\big(s,Y_s^m,Z_s^m,\textstyle\int_\Gamma U_s^m(e)\nu(de)\big)\Big|ds,$$

$$\tilde{J}_{2,t} := J_{2,t};\ \tilde{J}_{3,t} := J_{3,t} + \beta M_2\sqrt{\ln(A_N)}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}|\|\hat{U}_s^{n,m}\|_\nu ds,$$

and $\Phi(s) = |Y_s^n| + |Y_s^m| + |Z_s^n| + |Z_s^m| + \|U_s^n\|_\nu + \|U_s^m\|_\nu$.

Proof of Proposition 2. The proof closely aligns with the methodology employed in establishing Lemma A1. Let $C := C_N := 3\beta\frac{M_2^2}{\beta-1}\ln(A_N)$ and $\gamma := 3\delta\beta\frac{M_2^2}{\beta-1}$.

As presented in Lemma 9, it is obvious that

$$\begin{aligned}
&-\frac{C_N}{3}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds\\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds + \beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds\\
\le\ &\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N) - \frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2 + M_2\sqrt{\varphi_s}|\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{C_N}{3}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds - \beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds\\
&+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}|\|\hat{U}_s^{n,m}\|_\nu\sqrt{\ln(A_N)}ds\\
\le\ &\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N) - \frac{(\beta-1)}{2}\|\hat{U}_s^{n,m}\|_\nu^2\\
&\qquad + M_2\sqrt{\varphi_s}\|\hat{U}_s^{n,m}\|_\nu\sqrt{\ln(A_N)}\Big)ds.
\end{aligned}$$

Using Young's inequality, it follows that

$$-\frac{1}{\beta-1}M_2^2\ln(A_N)a^2 - \frac{(\beta-1)}{2}|b|^2 + M_2\,ab\sqrt{\ln(A_N)} \le -\frac{\beta-1}{4}b^2;$$

therefore,

$$\begin{aligned}
&-C_N\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds\\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds - \beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds + J_{3,t}\\
\le\ &-\beta\frac{(\beta-1)}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}(|\hat{Z}_s^{n,m}|^2 + \|\hat{U}_s^{n,m}\|_\nu^2)ds.
\end{aligned}$$

Based on the preceding lemmas, for any $N > R$ we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(S-\delta)^+\le t\le S}|\hat{Y}_t^{n,m}|^\beta + \mathbb{E}\int_{(S-\delta)^+}^{S}\frac{\left(|\hat{Z}_s^{n,m}|^2+\|\hat{U}_s^{n,m}\|_\nu^2\right)}{\left(|\hat{Y}_s^{n,m}|^2+\Lambda_R\right)^{\frac{2-\beta}{2}}}ds\Big]\\
\le\;& \frac{\ell}{\beta-1}e^{C_N\delta}\mathbb{E}\big[|\hat{Y}_S^{n,m}|^\beta\big]+\frac{\ell}{\beta-1}\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}}\\
&+\frac{4\ell}{\beta-1}\beta K_4^{\frac{\alpha}{2}}(4TK_2+T\Lambda_R)^{\frac{\beta-1}{2}}(8TK_2+16K_1+16K_3)^{\frac{\kappa}{2}}\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}}\\
&+\frac{2\ell}{\beta-1}e^{C_N\delta}\beta(2N^2+\Lambda_1)^{\frac{\beta-1}{2}}[\rho_N(f_n-f)+\rho_N(f_m-f)].
\end{aligned}$$

Since $\delta < \frac{\beta-1}{3M_2^2}\min(\frac{1}{2},\frac{\kappa}{r\beta})$, we proceed by taking limits for n and m, followed by a limit as N approaches infinity, in accordance with the statement (*iii*) of Lemma 8, and we obtain the desired result. □

Theorem 2. *Under Assumption 2 Equation (5) has a unique solution (Y, Z, U) in $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$.*

To prove the above theorem, we utilize Proposition 2 and follow similar steps in the proof of the existence and uniqueness parts of Theorem 1.

4. The Relationship Between BSDEJs and QBSDEJs

We present a supplementary BSDEJ, explicitly formulated through the exponential transformation of the initial problem. This formulation facilitates the establishment of a connection between the solution of the auxiliary BSDEJ and that of the original BSDEJ (ζ, g). Subsequently, we will demonstrate an application to quadratic BSDEJs.

Lemma 10 (General exponential transformation)**.** *We assume that either (ζ, g) or $(\widetilde{\zeta},\widetilde{g})$ satisfies the first Assumption 1. Let $h\in\mathbb{L}^1(\mathbb{R})$ a measurable function and $[u]_h(y)$, $J_u^h(y)$ two operators, defined as*

$$\begin{aligned}
[u]_h(y) &:= \int_\Gamma \frac{\Psi(y+u(e))-\Psi(y)-\Psi'(y)u(e)}{\Psi'(y)}\nu(de),\\
J_u^h(y) &:= \int_\Gamma (\Psi^{-1}(y+u(e))-\Psi^{-1}(y)-(\Psi^{-1})'(y)u(e))\nu(de),
\end{aligned}$$

where Ψ is defined for every $x\in\mathbb{R}$ as

$$\Psi(x) = \int_0^x \exp\left(2\int_0^y h(t)dt\right)dy.$$

The triplet (Y, Z, U) is a solution to the BSDEJ (ζ, g) if and only if the triplet $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ is a solution to the BSDEJ $(\widetilde{\zeta},\widetilde{g})$, where

$$\widetilde{Y}_t = \Psi(Y_t),\ \widetilde{\zeta} = \Psi(\zeta),\ \widetilde{Z}_t = \Psi'(Y_t)Z_t,\ \widetilde{U}_t(e) = \Psi(Y_{t-}+U_t(e))-\Psi(Y_{t-}),$$

and

$$\begin{aligned}
(\Psi^{-1})'(\widetilde{y})\widetilde{g}(t,\widetilde{y},\widetilde{z},\widetilde{u}) =\;& g\big(t,\Psi^{-1}(\widetilde{y}),\widetilde{z}(\Psi^{-1})'(\widetilde{y}),\Psi^{-1}(\widetilde{y}+\widetilde{u})-\Psi^{-1}(\widetilde{y})\big)\\
&-\widetilde{z}^2h(\Psi^{-1}(\widetilde{y}))\big((\Psi^{-1})'(\widetilde{y})\big)^2+J_{\widetilde{u}}^h(\widetilde{y}).
\end{aligned}$$

Clearly, Ψ is bi-Lipschitz with $\Psi(0)=0$, guaranteeing the preservation of the same spaces for primary BSDEJs and their auxiliary counterparts, i.e., (Y, Z, U) and $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ in

$\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R}) \times \mathbb{H}^2([0,T];\mathbb{R}) \times \mathbb{L}^2([0,T],\nu;\mathbb{R})$. The proof proceeds through a series of steps analogous to those outlined in Lemma 11.

Example 2. *Consider ζ satisfying condition (**A.1**), and let $g(t,y,z,u)$ be a continuous function with respect to (y,z,u). The function is defined as follows:*

$$
\begin{aligned}
g(t,y,z,u) &= \frac{1}{\Psi'(y)}\Big[\Psi(y)|\ln|\Psi(y)|| + z\Psi'(y)\sqrt{|\ln|z\Psi'(y)||} \\
&\quad + \|\Psi(y+u(e)) - \Psi(y)\|_\nu\Big] + h(y)|z|^2 + [u]_h(y),
\end{aligned}
$$

where Ψ is defined as in the previous Lemma 10. Using its result, it becomes evident that the BSDEJ (ζ, g) is equivalent to the BSDEJ $(e^\zeta, \tilde{y}|\ln|\tilde{y}|| + \tilde{z}\sqrt{|\ln|\tilde{z}||} + \|\tilde{u}\|_\nu)$, whose generator satisfies Assumption 1, and ensures the existence and uniqueness of the solution for both BSDEJ. Furthermore, (Y,Z,U), $(\tilde{Y},\tilde{Z},\tilde{U})$ in $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R}) \times \mathbb{H}^2([0,T];\mathbb{R}) \times \mathbb{L}^2([0,T],\nu;\mathbb{R})$.

Proposition 3. *Assuming that Assumption 1 holds and further supposing that ζ and $(\vartheta_t)_{0\le t\le T}$ are bounded, then there exists C_T such that*

- $\sup_{t\in[0,T]} |Y_t| \le C_T$.
- $\mathbb{E}[\int_0^T (|Z_s|^2 + \|U_s\|_\nu^2)ds] \le C_T$.

Proof. By utilizing Itô's formula and employing the same step as in the proof of Lemma 5, we obtain

$$
|Y_t|^{\mu_t+1} \le C + |\zeta|^{\mu_T+1} + \int_0^T (\mu_s+1)^{\mu_s+1}\vartheta_s^{\mu_s+1} ds + \mathbb{M}_t,
$$

where

$$
\mathbb{M}_t := -\int_t^T (\mu_s+1)|Y_s|^{\mu_s}\mathrm{sgn}(Y_s)Z_s dW_s - \int_t^T\int_\Gamma \Big(|Y_s|^{\mu_s+1} - |Y_{s-}|^{\mu_s+1}\Big)\tilde{N}(ds,de).
$$

We obtain the first result by taking the conditional expectation. Building upon the first result and condition (ii) in Lemma 5, we attain the desired outcome. □

Let $\lambda > 0$ and $t \in [0,T]$. Consider the following BSDEJ:

$$
\begin{aligned}
Y_t &= \zeta + \int_t^T \big(g(s,Y_s,Z_s,U_s) + \frac{\lambda}{2}|Z_s|^2 + [U_s]_\lambda\big)ds \\
&\quad - \int_t^T Z_s dW_s - \int_t^T\int_\Gamma U_s(e)\tilde{N}(ds,de),
\end{aligned} \tag{6}
$$

where

$$
[u]_\lambda = \frac{1}{\lambda}\int_\Gamma (e^{\lambda u(e)} - \lambda u(e) - 1)\nu(de).
$$

Assumption 3.

(**A.4**) (i) *The function g is continuous in (y,z) and Lipschitz with respect to u for almost all (t,ω).*

(ii) *There exist constants c_0, c_1, c_2, and C_{Lip}, as well as a bounded positive process $(\vartheta_t)_{t\ge 0}$, such that for every t, ω, y, z, u, u_1, u_2:*

$$
|g(t,y,z,u)| \le \vartheta_t + c_2|y| + c_0|z|\sqrt{|\ln|\lambda z| + \lambda y|} + \frac{c_1}{\lambda}\int_\Gamma (e^{\lambda u(e)} - 1)\nu(de),
$$

and

$$|g(t,\omega,y,z,u_1)-g(t,\omega,y,z,u_2)| \leq C_{Lip}\|u_1-u_2\|_\nu.$$

*(**A.5**) There exists a real-valued sequence $(A_N)_{N>1}$ and constants $M_2 \in \mathbb{R}_+$, $r>0$ such that*

(i) $\forall\, N>1,\quad 1<A_N\leq N^r$.

(ii) $\lim_{N\to\infty} A_N=\infty$.

(iii) *For every $N\in\mathbb{N}$, and every $y_1,\ y_2,\ z_1,\ z_2,\ u$ such that for all $|y_1|,\ |y_2|\leq \ln(N)$ $|z_1|,|z_2|\leq 1,\ u\leq \ln(2)$, we have*

$$\begin{aligned}&(e^{\lambda y_1}-e^{\lambda y_2})(e^{\lambda y_1}g(t,\omega,y_1,z_1,u)-e^{\lambda y_2}g(t,\omega,y_2,z_2,u))\\ \leq\ & M_2\Big(\,|\,e^{\lambda y_1}-e^{\lambda y_2}\,|^2\ln(A_N)\\ &\quad +|e^{\lambda y_1}-e^{\lambda y_2}||z_1e^{\lambda y_1}-z_2e^{\lambda y_2}|\sqrt{\ln(A_N)}+\frac{\ln(A_N)}{A_N}\Big).\end{aligned}$$

In the following lemma, we utilize the exponential transformation while relaxing the Lipschitz condition through the utilization of $\Psi(x)=e^{\lambda x}$.

Lemma 11. *If ζ and $(\vartheta_t)_{0\leq t\leq T}$ are bounded and Assumption 3 holds, then, for any $\lambda>0$, the following equivalence holds: there exists a unique solution*

$$(Y,Z,U)\in\mathcal{S}^\infty([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$$

to the BSDEJ (6) if and only if the triplet

$$(\widetilde{Y},\widetilde{Z},\widetilde{U})\in\mathcal{S}^\infty([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$$

is the unique solution to the BSDEJ$(\widetilde{\zeta},\widetilde{g})$, where

$$\widetilde{Y}_t = e^{\lambda Y_t},\ \widetilde{\zeta} = e^{\lambda\zeta},\ \widetilde{Z}_t = \lambda e^{\lambda Y_t}Z_t,\ \widetilde{U}_t = e^{\lambda Y_{t-}}(e^{\lambda U_t}-1),$$

and

$$\widetilde{g}(t,\widetilde{y},\widetilde{z},\widetilde{u}) = \lambda\widetilde{y}g\Big(t,\frac{1}{\lambda}\ln(\widetilde{y}),\frac{\widetilde{z}}{\lambda\widetilde{y}},\frac{1}{\lambda}\ln\Big(1+\frac{\widetilde{u}}{\widetilde{y}}\Big)\Big).$$

Proof. By employing Itô's formula on $\widetilde{Y}_t=e^{\lambda Y_t}$, we derive the following result for all $t\in[0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned}\widetilde{Y}_t &= \widetilde{\zeta}+\int_t^T \lambda e^{\lambda Y_s}g(s,Y_s,Z_s,U_s)ds\\ &\quad -\int_t^T\lambda e^{\lambda Y_s}Z_sdW_s-\int_t^T\int_\Gamma e^{\lambda Y_{s-}}(e^{\lambda U_s(e)}-1)\widetilde{N}(de,ds).\end{aligned}$$

With the quantities provided above, we can deduce the following:

$$\widetilde{Y}_t = \widetilde{\zeta}+\int_t^T\widetilde{g}(s,\widetilde{Y}_s,\widetilde{Z}_s,\widetilde{U}_s)ds-\int_t^T\widetilde{Z}_sdW_s-\int_t^T\int_\Gamma\widetilde{U}_s(e)\widetilde{N}(de,ds). \tag{7}$$

Since the generator g satisfies Assumption 3, then the generator $\widetilde{g}$ fulfills Assumption 1; therefore, Theorem 1 shows that Equation (7) has a unique solution in $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$. Thus, taking account of Proposition 3, the necessary condition is proved.

Conversely, Itô's formula applied to $\ln(\widetilde{Y}_t)/\lambda$ along with Proposition 3 lead to the sufficient condition.

It is worth mentioning that the functional spaces are conserved due to Proposition 3. □

Example 3. *Assume ζ is bounded, and let*

$$g(t,y,z,u) \quad = \quad c_2|y| + c_0|z|\sqrt{|\ln|\lambda z| + \lambda y|} + \frac{c_1}{\lambda}\|e^{\lambda u} - 1\|_\nu,$$

where c_0, c_1, and c_2 are positive constants. Therefore,

$$\widetilde{g}(t,\widetilde{y},\widetilde{z},\widetilde{u}) \quad = \quad c_2|\widetilde{y}||\ln|\widetilde{y}|| + c_0|\widetilde{z}|\sqrt{|\ln|\widetilde{z}||} + c_1\|\widetilde{u}\|_\nu.$$

Clearly, the generator $\widetilde{g}$ satisfies Assumption 1. Consequently, according to the preceding Lemma 11, the BSDEJ(ζ, g) has a unique solution and the BSDEJ$(\widetilde{\zeta}, \widetilde{g})$ has a unique solution.

Remark 1 (Quadratic–exponential BSDEJs)**.** *Let $g_1(t,y) = g(t,y,0,0)$, where g is defined as in the previous example. Then, the BSDEJ (6) transforms into a quadratic–exponential BSDEJ, which has a unique solution.*

For a more extensive examination of quadratic BSDEJs, we refer to [30].

Remark 2. *The primary BSDEJs discussed in the previous section share the same auxiliary counterpart, consistent with the discussions in this section regarding the suitable space for the jump. In other words, the previously established lemmas hold for the generators $g(s,y,z,\int_\Gamma u(e)\nu(de))$ and $\tilde{g}(s,\tilde{y},\tilde{z},\int_\Gamma \tilde{u}(e)\nu(de))$.*

5. Conclusions

Our study addresses fundamental questions concerning the existence and uniqueness of BSDEs whose driving processes are a compensated Poisson random measure and an independent Wiener process. Through rigorous proofs under two sets of assumptions, we first emphasize the significance of a generator by the logarithmic growth in both (y,z)-variables and the Lipschitz continuity with respect to the third variable u. We also included a concrete example that strengthens the validity of our first assumption.

Under Assumption 2, we take a step further by relaxing the Lipschitz condition on u. Here, the generator exhibits logarithmic growth in all variables, adding nuance to our understanding of the problem. Moreover, the introduction of the exponential transformation proves to be a key tool that demonstrates the equivalence between the solutions of the auxiliary BSDEJ and our primary BSDEJ.

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Appendix A

Proof of Lemma 1. We consider two cases based on the relationship between $|y|$ and $|z|$.

Case 1 : $|z| \leq |y|$

In this case, we have $1 < |y|\ln|y|$ and $\ln|z|\, \mathbb{1}_{\{|z|>1\}} \leq \ln|y|\, \mathbb{1}_{\{|z|>1\}}$, thus:

$$\begin{aligned} C_1|z||y|\sqrt{-\ln|z|}\, \mathbb{1}_{\{|z|\leq 1\}} &\leq e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}|y| \\ &\leq e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}|y|^2\ln|y| \\ &\leq \frac{|z|^2}{2} + e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}|y|^2\ln|y|, \end{aligned}$$

and

$$\begin{aligned} C_1|z||y|\sqrt{\ln|z|}\, \mathbb{1}_{\{|z|>1\}} &\leq \frac{|z|^2}{2} + 2C_1^2|y|^2\ln|z|\, \mathbb{1}_{\{|z|>1\}} \\ &\leq \frac{|z|^2}{2} + 2C_1^2|y|^2\ln|y|. \end{aligned}$$

The inequality (2) becomes

$$C_1|y||z|\sqrt{|\ln|z||} \leq \frac{|z|^2}{2} + C_2|y|^2\ln|y|,$$

where $C_2 = 2C_1^2 \vee e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}$. Therefore, the inequality holds in this case.

Case 2: $|z| > |y|$

Let us set $a = \frac{|z|}{|y|} > 1$. Since $|y| \geq e$, we have $|z| = a|y| > e$. Using this substitution, the inequality becomes

$$\begin{aligned} C_1|y||z|\sqrt{\ln|z|} &\leq C_1 a|y|^2\left(\sqrt{\ln(a)} + \sqrt{\ln|y|}\right) \\ &\leq |y|^2\left(\frac{a^2}{4} + C_1^2\ln|y| + C_1 a\sqrt{\ln(a)}\right); \end{aligned}$$

the latter inequality was derived from Young's inequality. Moreover, we have

$$\frac{|z|^2}{2} + C_2|y|^2\ln|y| = \left(\frac{a^2}{2} + C_2\ln|y|\right)|y|^2.$$

We obtain the desired result by showing that

$$\frac{a^2}{4} + C_1 a\sqrt{\ln(a)} + C_1^2\ln|y| \leq \frac{a^2}{2} + C_2\ln|y|.$$

Let $r = \max\{z \geq 1 : 4C_1\sqrt{\ln(z)} - z = 0\}$, and let us introduce the function h, defined as $h : t \in \mathbb{R}_+ \longrightarrow h(t) := 4C_1\sqrt{\ln(t)} - t$. We denote by $r_0 = \arg\max_{t>0} h(t)$; it follows that $r_0\sqrt{\ln(r_0)} = 2C_1$

There are two sub-cases to consider:

Sub-Case 1: If $C_1 \geq \frac{r_0}{4\sqrt{\ln(r_0)}}$, then r is well defined. If $a \geq r$, then $C_1 a\sqrt{\ln(a)} \leq \frac{a^2}{4}$, and if $1 < a < r$, then since $|y| \geq e$, we have

$$C_1 a\sqrt{\ln(a)} \leq C_1 r\sqrt{\ln(r)} = C_1\frac{r^2}{4} \leq C_2 \leq C_2\ln|y|.$$

Sub-Case 2: If $C_1 < \frac{r_0}{4\sqrt{\ln(r_0)}}$, since $2C_1 = r_0\sqrt{\ln(r_0)}$, then $r_0 < e^{\frac{1}{2}}$, which implies that $C_1 < \sqrt{2}e^{\frac{1}{2}}$. Therefore,

$$C_1 a\sqrt{\ln(a)} \;<\; \sqrt{2}e^{\frac{1}{2}}a\sqrt{\ln(a)} \;<\; \frac{a^2}{4} + 11 \;<\; \frac{a^2}{4} + 11\ln|y|, \;\; \text{since } |y| > e.$$

Therefore, the inequality holds in all cases, which completes the proof. □

Proof of Lemma 2. Let $y = 0$. In this case, the integral simplifies to $\int_0^1 (1-a)|x|^p da = \frac{1}{2}|x|^p$. Thus, we consider the scenario where $y \neq 0$ and define $a_0 := \frac{2|x|}{3|y|}$. For any $a \in [0, a_0] \cup [2a_0, \infty)$, it holds that

$$\frac{1}{3}|x| \;\leq\; ||x| - a|y|| \;\leq\; |x + ay|.$$

We proceed by analyzing three distinct cases:

(1) **Case 1:** $1 \leq a_0$. In this case, we have

$$\int_0^1 (1-a)|x+ay|^p da \geq \Big(\frac{1}{3}|x|\Big)^p \int_0^1 (1-a)da = \frac{1}{2}\Big(\frac{1}{3}|x|\Big)^p.$$

(2) **Case 2:** $\frac{1}{2} \leq a_0 < 1$. Here, we observe

$$\begin{aligned}\int_0^1 (1-a)|x+ay|^p da &\geq \int_0^{\frac{1}{2}} (1-a)|x+ay|^p da \geq \Big(\frac{1}{3}|x|\Big)^p \int_0^{\frac{1}{2}} (1-a)da \\ &= \frac{3}{8}\Big(\frac{1}{3}|x|\Big)^p.\end{aligned}$$

(3) **Case 3:** $a_0 < \frac{1}{2}$. In this scenario, we have

$$\begin{aligned}\int_0^1 (1-a)|x+ay|^p da &\geq \Big(\frac{1}{3}|x|\Big)^p \Big(\int_0^{a_0} (1-a)da + \int_{2a_0}^1 (1-a)da\Big) \\ &= \Big(\frac{1}{3}|x|\Big)^p \Big(\frac{3}{2}a_0^2 - a_0 + \frac{1}{2}\Big) \;\geq\; \frac{1}{3}\Big(\frac{1}{3}|x|\Big)^p.\end{aligned}$$

□

Proof of Lemma 3 under Assumption 1. Set $u(t,x) := |x|^{\mu_t+1}$ and $\mathrm{sgn}(x) := -\mathbb{1}_{\{x\leq 0\}} + \mathbb{1}_{\{x>0\}}$, then $u_t(t,x) = \theta\mu_t \ln|x||x|^{\mu_t+1}$, $u_x(t,x) = (\mu_t+1)|x|^{\mu_t}\mathrm{sgn}(x)$ and $u_{xx}(t,x) = \mu_t(\mu_t+1)|x|^{\mu_t-1}$. By utilizing Itô's formula to $u(t, Y_t)$

$$\begin{aligned}u(T,Y_T) &= u(t,Y_t) + \int_t^T u_s(s,Y_s)ds + \int_t^T u_x(s,Y_{s-})dY_s + \int_t^T u_{xx}(s,Y_{s-})d\langle Y\rangle_s \\ &\quad + \sum_{t\leq s\leq T}(u(s,Y_s) - u(s,Y_{s-}) - u_x(s,Y_{s-})\Delta Y_s) \\ &= u(t,Y_t) + \int_t^T u_s(s,Y_s)ds + \int_t^T u_x(s,Y_{s-})dY_s + \int_t^T u_{xx}(s,Y_s)|Z_s|^2 ds \\ &\quad + \int_t^T \int_\Gamma (u(s,Y_{s-}+U_s(e)) - u(s,Y_{s-}) - u_x(s,Y_{s-})U_s(e))N(ds,de) \\ &= u(t,Y_t) + \int_t^T u_s(s,Y_s)ds + \int_t^T u_{xx}(s,Y_s)|Z_s|^2 ds \\ &\quad - \int_t^T u_x(s,Y_{s-})f(s,Y_s,Z_s,U_s)ds \\ &\quad + \int_t^T u_x(s,Y_s)Z_s dW_s + \int_t^T \int_\Gamma (u(s,Y_{s-}+U_s(e)) - u(s,Y_{s-}))\widetilde{N}(ds,de) \\ &\quad + \int_t^T \int_\Gamma (u(s,Y_{s-}+U_s(e)) - u(s,Y_{s-}) - u_x(s,Y_{s-})U_s(e))\nu(de)ds. \qquad \text{(A1)}\end{aligned}$$

Setting

$$\begin{aligned}\Xi_t &= \int_0^t u_x(s,Y_s)Z_s dW_s + \int_0^t \int_\Gamma (u(s,Y_{s-}+U_s(e)) - u(s,Y_{s-}))\widetilde{N}(ds,de) \\ &= \int_0^t (\mu_s+1)|Y_s|^{\mu_s}\operatorname{sgn}(Y_s)Z_s dW_s \\ &\quad + \int_0^t \int_\Gamma \Big(|Y_{s-}+U_s(e)|^{\mu_s+1} - |Y_{s-}|^{\mu_s+1}\Big)\widetilde{N}(ds,de)\end{aligned}$$

For $n \geq 0$, define the stopping time τ_n as follows:

$$\begin{aligned}\tau_n := \inf\Big\{0 \leq t \leq T : \int_0^t \Big((\mu_s+1)|Y_s|^{\mu_s}Z_s\Big)^2 ds \ \vee \\ \int_0^t \int_\Gamma \Big(|Y_{s-}+U_s(e)|^{\mu_s+1} - |Y_{s-}|^{\mu_s+1}\Big)^2 \nu(de)ds \ \geq \ n\Big\}.\end{aligned}$$

Taking $t = t \wedge \tau_n$ and $T = T \wedge \tau_n$ in the equality (A1), we obtain

$$\begin{aligned}& |Y_{t\wedge\tau_n}|^{\mu_{t\wedge\tau_n}+1} + \frac{1}{2}\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)\mu_s|Y_s|^{\mu_s-1}|Z_s|^2 ds + \int_{t\wedge\tau_n}^{T\wedge\tau_n} \theta\mu_s|Y_s|^{\mu_s+1}\ln|Y_s| ds \\ = \ & |Y_{T\wedge\tau_n}|^{\mu_{T\wedge\tau_n}+1} + \int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s} f(s,Y_s,Z_s,U_s) ds \\ & - \int_{t\wedge\tau_n}^{T\wedge\tau_n} \int_\Gamma \Big(|Y_{s-}+U_s(e)|^{\mu_s+1} - |Y_{s-}|^{\mu_s+1} - (\mu_s+1)|Y_{s-}|^{\mu_s}\operatorname{sgn}(Y_{s-})U_s(e)\Big)\nu(de)ds \\ & + \Xi_{t\wedge\tau_n} - \Xi_{T\wedge\tau_n}\end{aligned} \tag{A2}$$

By Assumption (**A.2**)-(ii)

$$\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s} f(s,Y_s,Z_s,U_s) ds \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}I_1 &:= \int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)\vartheta_s|Y_s|^{\mu_s} ds, \\ I_2 &:= c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s+1}|\ln|Y_s|| ds, \\ I_3 &:= c_0\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s}|Z_s|\sqrt{|\ln|Z_s||} ds, \\ I_4 &:= c_1\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s}\|U_s\|_\nu ds.\end{aligned}$$

Estimation of I_1: Young's inequality yields ($|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$, for $p := \mu_s + 1$ and $q := \frac{\mu_s+1}{\mu_s}$) leads to

$$(\mu_s+1)\vartheta_s|Y_s|^{\mu_s} \leq (\mu_s+1)^{\mu_s}\vartheta_s^{\mu_s+1} + \frac{\mu_s}{\mu_s+1}|Y_s|^{\mu_s+1}$$

Hence,

$$
\begin{aligned}
I_1 &\leq \int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)^{\mu_s}\vartheta_s^{\mu_s+1}ds + \int_{t\wedge\tau_n}^{T\wedge\tau_n} \frac{\mu_s}{\mu_s+1}|Y_s|^{\mu_s+1}ds \\
&\leq \int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)^{\mu_s}\vartheta_s^{\mu_s+1}ds + \int_{t\wedge\tau_n}^{T\wedge\tau_n} |Y_s|^{\mu_s+1}ds \\
&\leq (\mu_T+1)^{\mu_T}\int_0^T \vartheta_s^{\mu_s+1}ds + \int_0^T |Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>e\}}ds + Te^{\mu_T+1} \\
&\leq (\mu_T+1)^{\mu_T}\int_0^T \vartheta_s^{\mu_s+1}ds + \int_0^T |Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds + Te^{\mu_T+1}.
\end{aligned}
$$

Estimation of I_2: Due to the presence of $|\ln|y||$, we split the integral of I_2 into two parts:

$$
\begin{aligned}
I_2 &\leq c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s}(-|Y_s|\ \ln|Y_s|)\ \mathbb{1}_{\{|Y_s|\leq 1\}}ds \\
&\quad + c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds \\
&\leq c_2e^{-1}\int_0^T(\mu_s+1)ds + c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds.
\end{aligned}
$$

Estimation of I_3: Using Lemma 1, there exists a constant $c_3>0$ such that

$$
c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|>e\}} \leq \frac{1}{4}|z|^2\ \mathbb{1}_{\{|y|>e\}} + c_3|y|^2\ln|y|\ \mathbb{1}_{\{|y|>e\}}.
$$

We have

$$
|z|\sqrt{|\ln|z||} \quad\leq\quad e^{-\frac{1}{2}}\frac{1}{\sqrt{2}} + |z|^{\frac{3}{2}}\ \mathbb{1}_{\{|z|>1\}} \tag{A3}
$$

Thus,

$$
\begin{aligned}
c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|\leq e\}} &\leq c_0e^{\frac{1}{2}}\frac{1}{\sqrt{2}} + c_0e|z|^{\frac{3}{2}}\ \mathbb{1}_{\{|z|>1\}}\ \mathbb{1}_{\{|y|\leq e\}} \\
&\leq \frac{1}{4}|z|^2\ \mathbb{1}_{\{|y|\leq e\}} + \widetilde{c}_0,
\end{aligned}
$$

where the last inequality is obtained by Young's inequality (for $p=\frac{4}{3}$ and $q=4$) and $\widetilde{c}_0 = c_0e^{\frac{1}{2}}\frac{1}{\sqrt{2}} + 3^3\frac{(c_0e)^4}{4}$. Therefore,

$$
\begin{aligned}
I_3 &\leq \hat{C}_1 + \frac{1}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Z_s|^2|Y_s|^{\mu_s-1}ds + c_3\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>e\}}ds \\
&\leq \hat{C}_1 + \frac{1}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Z_s|^2|Y_s|^{\mu_s-1}ds + c_3\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds,
\end{aligned}
$$

where $\hat{C}_1 = \widetilde{c}_0(\frac{\mu_T-1}{\theta}+T)e^{\mu_T-1}$.

Estimation of I_4: We observe that we can derive for any small $\varrho\in(0,\frac{2}{3^{\mu_T}}]$

$$
\begin{aligned}
c_1|y|\|u\|_\nu &\leq c_1^2\frac{1}{\varrho}|y|^2 + \frac{\varrho}{4}\|u\|_\nu^2 \\
&\leq c_1^2\frac{1}{\varrho}e^2 + c_1^2\frac{1}{\varrho}|y|^2\ln|y|\ \mathbb{1}_{\{|y|>e\}} + \frac{\varrho}{4}\|u\|_\nu^2;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
I_4 &\le \hat{C}_2 + \frac{c_1^2}{\varrho}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\,\mathbb{1}_{\{|Y_s|>e\}}ds + \frac{\varrho}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s-1}\|U_s\|_\nu^2 ds\\
&\le \hat{C}_2 + \frac{c_1^2}{\varrho}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\,\mathbb{1}_{\{|Y_s|>1\}}ds + \frac{\varrho}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s-1}\|U_s\|_\nu^2 ds,
\end{aligned}
$$

where $\hat{C}_2 = \frac{c_1^2}{\varrho}(\frac{\mu_T-1}{\theta}+T)e^{\mu_T+1}$. It remains to estimate

$$
I_5 := -\int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma\Big(|Y_s+U_s(e)|^{\mu_s+1} - |Y_s|^{\mu_s+1} - (\mu_s+1)|Y_s|^{\mu_s}\mathrm{sgn}(Y_s)U_s(e)\Big)\nu(de)ds.
$$

By Taylor's formula and Lemma 2, we have

$$
\begin{aligned}
&|y+u|^{\mu_s+1} - |y|^{\mu_s+1} - (\mu_s+1)|y|^{\mu_s}\mathrm{sgn}(y)u\\
=\;& \mu_s(\mu_s+1)u^2\int_0^1(1-a)|y+au|^{\mu_s-1}da \ge \mu_s(\mu_s+1)u^2 3^{-\mu_s}|y|^{\mu_s-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_5 &\le -\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)3^{-\mu_s}|Y_s|^{\mu_s-1}\int_\Gamma|U_s(e)|^2\nu(de)ds\\
&= -\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)3^{-\mu_s}|Y_s|^{\mu_s-1}||U_s||_\nu^2 ds.
\end{aligned}
$$

Since $3^{-\mu_s}\ge 3^{-\mu_T}$ and $\mu_s\ge 1$, then $\frac{\varrho}{2}\le \mu_s 3^{-\mu_s}$, which implies that

$$
\begin{aligned}
I_4+I_5 &\le \hat{C}_2 + \frac{c_1^2}{\varrho}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\,\mathbb{1}_{\{|Y_s|>1\}}ds\\
&\quad -\frac{1}{2}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)3^{-\mu_s}|Y_s|^{\mu_s-1}||U_s||_\nu^2 ds.
\end{aligned}
$$

and

$$
\frac{1}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)(1-\mu_s)|Y_s|^{\mu_s-1}|Z_s|^2 ds \le 0.
$$

Moreover, for $\theta \ge 2(\frac{c_1^2}{\varrho}+c_2+c_3)+1$, we have $1+(\mu_s+1)(\frac{c_1^2}{\varrho}+c_2+c_3-\theta\mu_s)\le 0$, which yields to

$$
\begin{aligned}
&-\theta\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s|Y_s|^{\mu_s+1}\ln|Y_s|ds + \int_{t\wedge\tau_n}^{T\wedge\tau_n}\Big(1+(\mu_s+1)\Big(\frac{c_1^2}{\varrho}+c_2+c_3\Big)\Big)|Y_s|^{\mu_s+1}\ln|Y_s|\,\mathbb{1}_{\{|Y_s|>1\}}ds\\
&= \int_{t\wedge\tau_n}^{T\wedge\tau_n}\Big(1+(\mu_s+1)(\frac{c_1^2}{\varrho}+c_2+c_3-\theta\mu_s)\Big)|Y_s|^{\mu_s+1}\ln|Y_s|\,\mathbb{1}_{\{|Y_s|>1\}}ds\\
&+\theta\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s|Y_s|^{\mu_s+1}(-\ln|Y_s|)\,\mathbb{1}_{\{|Y_s|\le 1\}}ds\\
&\le \theta\sup_{0<a\le 1}a(-\ln(a))\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s ds \;=\; \theta e^{-1}\int_0^T\mu_s ds.
\end{aligned}
$$

By Equation (A2) and the preceding result, and noting that for any $0 \leq s \leq T, 3^{-\mu_T} \leq 3^{-\mu_s}$, it becomes evident that

$$\begin{aligned} & |Y_{t\wedge\tau_n}|^{\mu_{t\wedge\tau_n}+1}+\frac{1}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}|Z_s|^2ds \\ & +\frac{3^{-\mu_T}}{2}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}||U_s||_\nu^2ds \\ \leq\ & |Y_{T\wedge\tau_n}|^{\mu_{T\wedge\tau_n}+1}+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds-\Xi_{T\wedge\tau_n}+\Xi_{t\wedge\tau_n}+\hat{C}+C_1. \end{aligned} \tag{A4}$$

where $C_1 = 2e^{-1}(\mu_T - 1) + c_2Te^{-1}$ and $\hat{C} = \hat{C}_1 + \hat{C}_2 + Te^{\mu_T+1}$. Thus, we obtain

$$\begin{aligned} & \mathbb{E}\Big[|Y_{t\wedge\tau_n}|^{\mu_{t\wedge\tau_n}+1}+\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}(|Z_s|^2+||U_s||_\nu^2)ds\Big] \\ \leq\ & C\mathbb{E}\Big[1+|Y_{T\wedge\tau_n}|^{\mu_{T\wedge\tau_n}+1}+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds\Big]. \end{aligned}$$

By Fatou's lemma, we can pass to the limit as $n \to \infty$. Consequently, the desired result follows. □

Proof of Lemma 4 under (A.1) and (A.2). Letting $\alpha \in (1, 2)$, we have

$$\begin{aligned} |y||\ln|y|| &\leq e^{-1}+|y|\ln|y|\ \mathbb{1}_{\{|y|>1\}} \\ &= e^{-1}+\frac{1}{\alpha-1}|y|\ln|y|^{\alpha-1}\ \mathbb{1}_{\{|y|>1\}} \\ &\leq e^{-1}+\frac{1}{\alpha-1}|y|^\alpha\ \mathbb{1}_{\{|y|>1\}}, \end{aligned}$$

$$\begin{aligned} |z|\sqrt{|\ln|z||} &\leq \frac{e^{-\frac{1}{2}}}{\sqrt{2}}+|z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|z|>1\}} \\ &= \frac{e^{-\frac{1}{2}}}{\sqrt{2}}+\frac{1}{\sqrt{2(\alpha-1)}}|z|\sqrt{\ln|z|^{2(\alpha-1)}}\ \mathbb{1}_{\{|z|>1\}} \\ &\leq \frac{e^{-\frac{1}{2}}}{\sqrt{2}}+\frac{1}{\sqrt{2(\alpha-1)}}|z|^\alpha\ \mathbb{1}_{\{|z|>1\}}, \end{aligned}$$

and

$$\vartheta_t + c_1\|u\|_\nu \leq 1 + c_1 + \vartheta_t^\alpha + c_1\|u\|_\nu^\alpha.$$

Therefore, by (**A.2**)-(ii),

$$\begin{aligned} |f(s,\omega,y,z,u)| &\leq \vartheta_s+c_2|y||\ln|y||+c_0|z|\sqrt{|\ln|z||}+c_1\|u\|_\nu \\ &\leq \widetilde{c}(1+\vartheta_s^\alpha+|y|^\alpha+|z|^\alpha+\|u\|_\nu^\alpha), \end{aligned}$$

where $\widetilde{c}$ is a positive constant depending on c_0, c_1, c_2, and α. For any $p \geq 1$, $n \in \mathbb{N}$ with $n \geq 2$ and $(b_i)_{i\in\mathbb{N}} \in \mathbb{R}_+$, we have

$$(\sum_{i=1}^n b_i)^p \leq n^{p-1}\sum_{i=1}^n b_i^p.$$

Thus,

$$\begin{aligned} |f(s,\omega,y,z,u)|^{\frac{2}{\alpha}} &\leq \widetilde{c}^{\frac{2}{\alpha}}(1+\vartheta_s^\alpha+|y|^\alpha+|z|^\alpha+\|u\|_\nu^\alpha)^{\frac{2}{\alpha}} \\ &\leq \widetilde{c}^{\frac{2}{\alpha}}5^{\frac{2-\alpha}{\alpha}}(1+\vartheta_s^2+|y|^2+|z|^2+\|u\|_\nu^2). \end{aligned}$$

Since $|y|^2 \leq 1 + |y|^{\mu_s+1}$, we can derive a positive constant $C(T, \alpha, c_0, c_1, c_2)$, such that

$$\int_0^T \mathbb{E}\big[|f(s, Y_s, Z_s, U_s)|^{\frac{2}{\alpha}}\big]ds \quad \leq \quad C(T, \alpha, c_0, c_1, c_2)\bigg(1 + \int_0^T \mathbb{E}\big[\vartheta_s^2 + |Y_s|^{\mu_s+1} + |Z_s|^2 + \|U_s\|_\nu^2\big]ds\bigg).$$

□

Proof of Lemma 5 under Assumption 1. We begin by proving assertion (i), which relies on Lemma 3.

For $n \geq 0$, define the stopping time $\widetilde{\tau}_n$ as follows:

$$\widetilde{\tau}_n := \inf\{s \geq 0 \ : \ |Y_s|^{\mu_s+1} > n\}.$$

By taking the same steps as in the previous proof of Lemma 3, we obtain the inequality (A4) for $\widetilde{\tau}_n$

$$\begin{aligned} & |Y_{t\wedge\widetilde{\tau}_n}|^{\mu_{t\wedge\widetilde{\tau}_n}+1} + \frac{1}{4}\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} \mu_s(\mu_s+1)|Y_s|^{\mu_s-1}|Z_s|^2 ds \\ & + \frac{3^{-\mu_T}}{2}\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} \mu_s(\mu_s+1)|Y_s|^{\mu_s-1}||U_s||_\nu^2 ds \\ \leq \quad & |Y_{T\wedge\widetilde{\tau}_n}|^{\mu_{T\wedge\widetilde{\tau}_n}+1} + (\mu_T+1)^{\mu_T}\int_0^T \vartheta_s^{\mu_s+1}ds - \Xi_{T\wedge\widetilde{\tau}_n} + \Xi_{t\wedge\widetilde{\tau}_n} + C, \end{aligned}$$

where C is a generic positive constant that may vary. Thus, we have

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n} |Y_t|^{\mu_t+1}\Big] \quad \leq \quad & C\Big(1 + \mathbb{E}\Big[|Y_{T\wedge\widetilde{\tau}_n}|^{\mu_T+1} + (\mu_T+1)^{\mu_T}\int_0^T \vartheta_s^{\mu_s+1}ds\Big]\Big) \\ & + \mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}\Big|\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} d\Xi_s\Big|\Big]. \end{aligned} \tag{A5}$$

Consider the following inequality, which holds for any non-negative $a, b \geq 0$ and $p > 1$,

$$|a^p - b^p| \leq p(a \vee b)^{p-1}|a - b|.$$

Therefore,

$$||Y_{s-} + U_s(e)|^{\mu_s+1} - |Y_{s-}|^{\mu_s+1}| \quad \leq \quad (\mu_s+1)(|Y_{s-} + U_s(e)| \vee |Y_{s-}|)^{\mu_s}|U_s(e)|,$$

clearly, $\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n} |Y_{t-}|^{\mu_t+1} \leq \sup_{0\leq t\leq T\wedge\widetilde{\tau}_n} |Y_t|^{\mu_t+1}$ and since $Y_s = Y_{s-} + U_s(e)$, then,

$$\begin{aligned} & ||Y_{s-} + U_s(e)|^{\mu_s+1} - |Y_{s-}|^{\mu_s+1}|^2 \\ \leq \quad & (\mu_s+1)^2(|Y_{s-} + U_s(e)| \vee |Y_{s-}|)^{2\mu_s}|U_s(e)|^2 \\ \leq \quad & (\mu_s+1)^2 \sup_{0\leq t\leq T\wedge\widetilde{\tau}_n} |Y_t|^{\mu_t+1}(|Y_{s-} + U_s(e)| \vee |Y_{s-}|)^{\mu_s-1}|U_s(e)|^2, \end{aligned}$$

Moreover, we have $(\mu_s+1)^2 < 3\mu_s(\mu_s+1)$. Applying Burkholder–Davis–Gundy inequality to $\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} d\Xi_s$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T\wedge\widetilde{\tau}_n}\Big|\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} d\Xi_s\Big|\Big]\\
\le\ & C\mathbb{E}\Big[\Big(\int_0^{T\wedge\widetilde{\tau}_n}(\mu_s+1)^2|Y_s|^{2\mu_s}|Z_s|^2ds\Big)^{\frac{1}{2}}\Big]\\
&+C\mathbb{E}\Big[\Big(\int_0^{T\wedge\widetilde{\tau}_n}\int_\Gamma\Big(|Y_{s-}+U_s(e)|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}\Big)^2N(ds,de)\Big)^{\frac{1}{2}}\Big]\\
\le\ & C\mathbb{E}\Big[\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\frac{\mu_t+1}{2}}\Big(\int_0^{T\wedge\widetilde{\tau}_n}(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big)^{\frac{1}{2}}\Big]\\
&+C\mathbb{E}\Big[\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\frac{\mu_t+1}{2}}\Big(\int_0^{T\wedge\widetilde{\tau}_n}\int_\Gamma(\mu_s+1)^2(|Y_{s-}+U_s(e)|\vee|Y_{s-}|)^{\mu_s-1}|U_s(e)|^2N(ds,de)\Big)^{\frac{1}{2}}\Big]\\
\le\ & \mathbb{E}\Big[\frac{1}{2}\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+C\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&+C\mathbb{E}\Big[\int_0^T\int_\Gamma(\mu_s+1)^2(|Y_{s-}+U_s(e)|\vee|Y_{s-}|)^{\mu_s-1}|U_s(e)|^2N(ds,de)\Big]
\end{aligned}$$

The last inequality is derived from Young's inequality ($ab\le\frac{1}{2}a^2+\frac{1}{2}b^2$), and the terms can be controlled as follows:

$$\begin{aligned}
=\ & \mathbb{E}\Big[\frac{1}{2}\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+C\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&+C\mathbb{E}\Big[\int_0^T\int_\Gamma(\mu_s+1)^2|Y_s|^{\mu_s-1}|U_s(e)|^2\nu(de)ds\Big]\\
=\ & \mathbb{E}\Big[\frac{1}{2}\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+C\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&+C\mathbb{E}\Big[\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds\Big]\\
\le\ & \mathbb{E}\Big[\frac{1}{2}\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+3C\int_0^T\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&+3C\mathbb{E}\Big[\int_0^T\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds\Big]\\
\le\ & \frac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}\Big]+C\mathbb{E}\Big[1+|\zeta|^{\mu_T+1}+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds\Big],
\end{aligned}$$

the last inequality is derived from Lemma 3. Observing that for any $n\ge 0$ we have $\widetilde{\tau}_n\le\widetilde{\tau}_{n+1}$, then $\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}\le\sup_{0\le t\le T\wedge\widetilde{\tau}_{n+1}}|Y_t|^{\mu_t+1}$. Consequently, by (A5) and by using the monotone convergence theorem, we obtain

$$\mathbb{E}[\sup_{0\le t\le T}|Y_t|^{\mu_t+1}]\quad\le\quad C\Big(1+\mathbb{E}[|\zeta|^{\mu_T+1}]+(\mu_T+1)^{\mu_T}\int_0^T\mathbb{E}[\vartheta_s^{\mu_s+1}]ds\Big).$$

This ends the proof of assertion (i).

We now advance to establish assertion (ii). The application of Itô's formula reveals that

$$\begin{aligned}
|Y_0|^2+\int_0^T(|Z_s|^2+\|U_s\|_\nu^2)ds+\Xi_T\ &=\ |\zeta|^2+2\int_0^T Y_sf(s,Y_s,Z_s,U_s)ds\\
&\le\ |\zeta|^2+2\int_0^T|Y_s|\big(\vartheta_s+g_{1,c_2}(Y_s)\big)ds\\
&\quad+2\int_0^T|Y_s|\big(g_{2,c_0}(Z_s)+c_1\|U_s\|_\nu\big)ds,
\end{aligned}$$

where $\Xi_t = 2\int_0^t Y_s Z_s dW_s + \int_0^t \int_\Gamma \left(2Y_{s-}U_s(e) + |U_s(e)|^2\right)\widetilde{N}(ds, de)$

For any given $\varepsilon > 0$, we have

$$
\begin{aligned}
|y|^2 |\ln |y|| &\leq -|y| \ln |y| \, \mathbb{1}_{\{|y \leq 1\}} + |y|^{2+\varepsilon} \, \mathbb{1}_{\{|y| > 1\}} \\
&\leq e^{-1} + |y|^{2+\varepsilon},
\end{aligned}
$$

and

$$
|y|^2 \leq |y|^{2+\varepsilon} \, \mathbb{1}_{\{|y| > 1\}} + 1.
$$

Furthermore, by Lemma 1 and employing Young's inequality, we can derive a positive constant $\widetilde{c}$, such that

$$
2c_0 |y||z| \sqrt{|\ln |z||} \, \mathbb{1}_{\{|y| > e\}} \leq \left(\frac{|z|^2}{2} + \widetilde{c}|y|^{2+\varepsilon}\right) \mathbb{1}_{\{|y| > e\}}.
$$

On the other hand, according to (A3)

$$
\begin{aligned}
2c_0 |y||z| \sqrt{|\ln |z||} \, \mathbb{1}_{\{|y| \leq e\}} &\leq 2c_0 e^{\frac{1}{2}} \frac{1}{\sqrt{2}} + 2c_0 e |z|^{\frac{3}{2}} \, \mathbb{1}_{\{|z| > 1\}} \, \mathbb{1}_{\{|y| \leq e\}} \\
&\leq \frac{1}{2} |z|^2 \, \mathbb{1}_{\{|y| \leq e\}} + \widetilde{c}_0,
\end{aligned}
$$

where $\widetilde{c}_0 = c_0 \sqrt{2} e^{\frac{1}{2}} + 4(c_0 e)^4 \left(\frac{3}{2}\right)^3$. By Young's inequality, we have

$$
2c_1 |y| \|u\|_\nu \, \mathbb{1}_{\{|y| > 1\}} \leq \left(\frac{\|u\|_\nu^2}{2} + 2c_1^2 |y|^{2+\varepsilon}\right) \mathbb{1}_{\{|y| > 1\}},
$$

$$
2c_1 |y| \|u\|_\nu \, \mathbb{1}_{\{|y| \leq 1\}} \leq \frac{\|u\|_\nu^2}{2} \, \mathbb{1}_{\{|y| \leq 1\}} + 2c_1^2.
$$

and

$$
2|y|\vartheta \leq \vartheta^2 + |y|^{2+\varepsilon} \, \mathbb{1}_{\{|y| > 1\}} + 1.
$$

Therefore,

$$
\begin{aligned}
\int_0^T \mathbb{E}\left[|Z_s|^2 + \|U_s\|_\nu^2\right] ds &\leq \widetilde{C}(T, c_0, c_1, c_2) \left(\hat{C} + \mathbb{E}\left[|\zeta|^2 + \int_0^T \vartheta_s^2 ds + \int_0^T |Y_s|^{2+\varepsilon} ds\right]\right) \\
&\leq \widetilde{C}(T, c_0, c_1, c_2) \left(\hat{C} + \mathbb{E}\left[|\zeta|^2 + \int_0^T \vartheta_s^2 ds + T \sup_{0 \leq t \leq T} |Y_t|^{2+\varepsilon}\right]\right).
\end{aligned}
$$

By selecting ε as $\mu_s - 1$, setting $t = 0$, and defining $C(T, c_0, c_1, c_2) = \widetilde{C}(T, c_0, c_1, c_2)(\hat{C} \vee 1)$, we obtain the desired result. □

Proof of Lemma 5 under Assumption 2. Consider a solution (Y, Z, U) to (5), and assume that conditions $(\mathbf{A.1})'$ and $(\mathbf{A.2})'$ are satisfied. We define the sign function $\text{sgn}(x)$ as follows: $\text{sgn}(x) = -1$ for $x \leq 0$ and $\text{sgn}(x) = +1$ for $x > 0$. We can apply Itô's formula to obtain

$$\begin{aligned}
|Y_{t\wedge\tilde{\tau}_n}|^{\mu_{t\wedge\tilde{\tau}_n}+1} \leq\ & |Y_{T\wedge\tilde{\tau}_n}|^{\mu_{T\wedge\tilde{\tau}_n}+1} + \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} (\mu_s+1)^{\mu_s} \vartheta_s^{\mu_s+1} ds + C_2 \\
& - \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} \theta\mu_s |Y_s|^{\mu_s+1} \ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}} ds \\
& + \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} |Y_s|^{\mu_s+1} \ln|Y_s|\ \mathbb{1}_{\{|Y_s|>e\}} ds \\
& + \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} (\mu_s+1)|Y_s|^{\mu_s} \Big(\frac{1}{2} g_{1,c_2}(Y_s)\ \mathbb{1}_{\{|Y_s|>1\}} + g_{2,c_0}(Z_s)\Big) ds \\
& + \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} (\mu_s+1)|Y_s|^{\mu_s} \Big(\frac{1}{2} g_{1,c_2}(Y_s)\ \mathbb{1}_{\{|Y_s|>1\}} + g_{3,c_1}(U_s)\Big) ds \\
& - \frac{1}{2} \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} (\mu_s+1)\mu_s |Z_s|^2 \mid Y_s \mid^{\mu_s-1} ds \\
& - \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} \mu_s(\mu_s+1) 3^{-\mu_s} |Y_s|^{\mu_s-1} ||U_s||_\nu^2 ds + \Xi_{t\wedge\tilde{\tau}_n} - \Xi_{T\wedge\tilde{\tau}_n}.
\end{aligned}$$

By Lemma 1, we have

$$c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|>e\}} \leq \frac{|z|^2}{4}\ \mathbb{1}_{\{|y|>e\}} + c_3|y|^2 \ln|y|\ \mathbb{1}_{\{y>e\}},$$

and

$$c_1|y|\|u\|_\nu \sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|>e\}} \leq \frac{\varrho}{4}\|u\|_\nu^2\ \mathbb{1}_{\{|y|>e\}} + c_4|y|^2 \ln|y|\ \mathbb{1}_{\{|y|>e\}}.$$

Utilizing Young's inequality, we obtain

$$c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|\leq e\}} \leq \frac{|z|^2}{4}\ \mathbb{1}_{\{|y|\leq e\}} + \tilde{c}_0,$$

and

$$c_1|y|\|u\|_\nu \sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|\leq e\}} \leq \frac{\varrho}{4}\|u\|_\nu^2\ \mathbb{1}_{\{|y|\leq e\}} + \tilde{c}_1.$$

where $\tilde{c}_0 = c_0 e^{\frac{1}{2}} \frac{1}{\sqrt{2}} + 3^3 \frac{(c_0 e)^4}{4}$, $\tilde{c}_1 = c_1 e^{\frac{1}{2}} \frac{1}{\sqrt{2}} + 3^3 \frac{(c_1 e)^4}{4}$. For $\theta \geq 2(c_2+c_3+c_4)+1$ we have $-\theta\mu_s + (c_2+c_3+c_4)(\mu_s+1)+1 \leq 0$, thus,

$$\int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} (-\theta\mu_s + (\mu_s+1)(c_2+c_3+c_4)+1)|Y_s|^{\mu_s+1} \ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}} ds \leq 0.$$

Thus, employing the same steps as outlined above, we can determine a general constant C such that

$$\mathbb{E}\big[\sup_{0\leq t\leq T\wedge\tilde{\tau}_n} |Y_{t\wedge\tilde{\tau}_n}|^{\mu_{t\wedge\tilde{\tau}_n}+1}\big] \leq C\mathbb{E}\Big[1 + |Y_{T\wedge\tilde{\tau}_n}|^{\mu_{T\wedge\tilde{\tau}_n}+1} + (\mu_T+1)^{\mu_T} \int_{t\wedge\tilde{\tau}_n}^{T\wedge\tilde{\tau}_n} \vartheta_s^{\mu_s+1} ds\Big],$$

The monotone convergence theorem enables us to obtain the assertion (i).

Since

$$2c_1|y|\ \|u\|_\nu \sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|>e\}} \leq \frac{\|u\|_\nu^2}{2}\ \mathbb{1}_{\{|y|>e\}} + \tilde{c}|y|^{2+\varepsilon},$$

and

$$2c_1|y|\ \|u\|_\nu \sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|\leq e\}} \leq \frac{\|u\|_\nu^2}{2}\ \mathbb{1}_{\{|y|\leq e\}} + \tilde{c}_1,$$

where $\tilde{c}_1 = c_1\sqrt{2}e^{\frac{1}{2}} + 4(c_1 e)^4 \left(\frac{3}{2}\right)^3$, we easily verify the validity of (ii). □

Lemma A1. *Assuming that the conditions of Proposition 1 are met, and defining φ_t as $|\hat{Y}^{n,m}_t|^2 + (A_N)^{-1}$, and $\kappa := 3 - \alpha - \beta$, we can establish the following result for any $C > 0$:*

$$\begin{aligned} e^{Ct}\varphi_t^{\frac{\beta}{2}} + C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \widetilde{\mathbb{M}}_t \quad \leq \quad & e^{CS}\varphi_S^{\frac{\beta}{2}} - \frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}^{n,m}_s|^2ds \\ & -\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}^{n,m}_s\|^2_\nu ds \\ & +\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}\,|\hat{Y}^{n,m}_s|^2|\hat{Z}^{n,m}_s|^2ds \\ & +\mathbb{M}_t + J_{1,t} + J_{2,t} + J_{3,t} + J_{4,t}, \end{aligned}$$

where

$$\begin{aligned} \widetilde{\mathbb{M}}_t &:= \int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}} - \varphi_{s-}^{\frac{\beta}{2}})\widetilde{N}(ds,de), \\ \mathbb{M}_t &:= -\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}^{n,m}_s\hat{Z}^{n,m}_s dW_s, \\ J_{1,t} &:= \beta e^{CS}\frac{1}{N^\kappa}\int_t^S \varphi_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)|f_n(s,Y^n_s,Z^n_s,U^n_s) - f_m(s,Y^m_s,Z^m_s,U^n_s)|ds, \\ J_{2,t} &:= \beta e^{CS}[4N^2+\Lambda_1]^{\frac{\beta-1}{2}}\Bigg[\int_t^S \sup_{|y|,|z|,\|u\|_\nu\leq N}|(f_n-f)(s,y,z,u)|ds \\ &\qquad + \int_t^S \sup_{|y|,|z|,\|u\|_\nu\leq N}|(f_m-f)(s,y,z,u)|ds\Bigg], \\ J_{3,t} &:= \beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(\varphi_s\ln(A_N) + \sqrt{\ln(A_N)}|\hat{Y}^{n,m}_s||\hat{Z}^{n,m}_s|\Big)ds, \\ J_{4,t} &:= \beta C_{Lip}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}^{n,m}_s|\|\hat{U}^{n,m}_s\|_\nu ds, \end{aligned}$$

and $\Phi(s) = |Y^n_s| + |Y^m_s| + |Z^n_s| + |Z^m_s| + \|U^n_s\|_\nu + \|U^m_s\|_\nu$.

Proof of Lemma A1. Let $C > 0$. For any positive integer N, we define the function $u(s,y)$ as

$$u(s,y) \quad = \quad e^{Cs}(\theta(y))^{\frac{\beta}{2}},$$

where $\theta(y) := y^2 + (A_N)^{-1}$; this yields the following partial derivatives:

$$\begin{aligned} u_s(s,y) \quad &= \quad Cu(s,y);\;\; u_y(s,y) = \beta e^{Cs}y(\theta(y))^{\frac{\beta}{2}-1}, \\ u_{yy}(s,y) \quad &= \quad \beta e^{Cs}(\theta(y))^{\frac{\beta}{2}-1} + \beta(\beta-2)e^{Cs}y^2(\theta(y))^{\frac{\beta}{2}-2}. \end{aligned}$$

Since $1 < \beta < 2$, we can establish that

$$u_{yy}(s,y) \quad \geq \quad \beta(\beta-1)e^{Cs}(\theta(y))^{\frac{\beta}{2}-1}.$$

Consequently, for all $s \in [0,T]$, we obtain, by Taylor expansion, that

$$\begin{aligned} & u(s,\hat{Y}^{n,m}_s) - u(s,\hat{Y}^{n,m}_{s-}) - \hat{U}^{n,m}_s(e)u_y(s,\hat{Y}^{n,m}_{s-}) \\ = \quad & |\hat{U}^{n,m}_s(e)|^2\int_0^1(1-a)u_{yy}(s,a\hat{U}^{n,m}_s(e) + \hat{Y}^{n,m}_{s-})da \\ \geq \quad & \beta(\beta-1)e^{Cs}|\hat{U}^{n,m}_s(e)|^2\int_0^1(1-a)\big(\theta(a\hat{U}^{n,m}_s(e) + \hat{Y}^{n,m}_{s-})\big)^{\frac{\beta}{2}-1}da. \end{aligned}$$

Since $0 \leq a \leq 1$, we have

$$\begin{aligned}\theta(a\hat{U}_s^{n,m}(e)+\hat{Y}_{s-}) &= |a\hat{U}_s^{n,m}(e)+\hat{Y}_{s-}^{n,m}|^2+(A_N)^{-1}\\ &= |a(\hat{Y}_{s-}^{n,m}+\hat{U}_s^{n,m}(e))+(1-a)\hat{Y}_{s-}^{n,m}|^2+(A_N)^{-1}\\ &\leq (|\hat{Y}_{s-}^{n,m}|\vee|\hat{Y}_s^{n,m}|)^2+(A_N)^{-1}.\end{aligned}$$

Given that $\frac{\beta}{2}-1$ is negative, hence

$$\left(\theta(a\hat{U}_s^{n,m}(e)+\hat{Y}_{s-}^{n,m})\right)^{\frac{\beta}{2}-1} \geq \left((|\hat{Y}_{s-}^{n,m}|\vee|\hat{Y}_s^{n,m}|)^2+(A_N)^{-1}\right)^{\frac{\beta}{2}-1}.$$

Therefore,

$$\begin{aligned}&u(s,\hat{Y}_s^{n,m})-u(s,\hat{Y}_{s-}^{n,m})-\hat{U}_s^{n,m}(e)u_y(s,\hat{Y}_{s-}^{n,m})\\ \geq\ &\beta(\beta-1)e^{Cs}|\hat{U}_s^{n,m}(e)|^2\int_0^1(1-a)\left(\theta(a\hat{U}_s^{n,m}(e)+\hat{Y}_{s-}^{n,m})\right)^{\frac{\beta}{2}-1}da\\ \geq\ &\beta\frac{(\beta-1)}{2}e^{Cs}|\hat{U}_s^{n,m}(e)|^2\left((|\hat{Y}_{s-}^{n,m}|\vee|\hat{Y}_s^{n,m}|)^2+(A_N)^{-1}\right)^{\frac{\beta}{2}-1}.\end{aligned} \tag{A6}$$

Applying Itô's formula to $u(t,Y_t)$ reveals that

$$\begin{aligned}&e^{Ct}\varphi_t^{\frac{\beta}{2}}+C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds\\ =\ &e^{CS}\varphi_S^{\frac{\beta}{2}}+\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}(f_n(s,Y_s^n,Z_s^n,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^m))ds\\ &-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds\\ &-\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\hat{Z}_s^{n,m}dW_s\\ &-\int_t^S e^{Cs}\int_\Gamma\left(\left(|\hat{Y}_{s-}^{n,m}+\hat{U}_s^{n,m}(e)|^2+(A_N)^{-1}\right)^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}}-\beta\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\right)N(ds,de)\\ &-\beta\int_t^S e^{Cs}\int_\Gamma\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\widetilde{N}(ds,de).\end{aligned}$$

By (A6), we can reformulate the jump components as follows:

$$\begin{aligned}&-\beta\int_t^S\int_\Gamma e^{Cs}\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\widetilde{N}(ds,de)\\ &-\int_t^S\int_\Gamma e^{Cs}\left(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}}-\beta\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\right)N(ds,de)\\ =\ &-\int_t^S\int_\Gamma e^{Cs}\left(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}}-\beta\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\right)\nu(de)ds\\ &-\int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}})\widetilde{N}(ds,de)\\ \leq\ &-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\|\hat{U}_s^{n,m}\|_\nu^2\left((|\hat{Y}_{s-}^{n,m}|\vee|\hat{Y}_s^{n,m}|)^2+(A_N)^{-1}\right)^{\frac{\beta}{2}-1}ds\\ &-\int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}})\widetilde{N}(ds,de)\\ \leq\ &-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2ds-\int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}})\widetilde{N}(ds,de).\end{aligned}$$

Therefore,

$$\begin{aligned} e^{Ct}\varphi_t^{\frac{\beta}{2}} + C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \widetilde{\mathbb{M}}_t \quad \leq \quad & e^{CS}\varphi_S^{\frac{\beta}{2}} - \frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds \\ & -\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds \\ & +\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\ & +\mathbb{M}_t + \acute{J}_{1,t} + \acute{J}_{2,t} + \acute{J}_{3,t} + \acute{J}_{4,t} + \acute{J}_{5,t}, \end{aligned}$$

where

$$\widetilde{\mathbb{M}}_t := \int_t^S \int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}} - \varphi_{s-}^{\frac{\beta}{2}})\widetilde{N}(ds, de),$$

and

$$\mathbb{M}_t := -\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\hat{Z}_s^{n,m}dW_s,$$

are $\mathbb{F}$-martingales, and

$$\begin{aligned} \acute{J}_{1,t} &:= \beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}(f_n(s,Y_s^n,Z_s^n,U_s^n) - f_m(s,Y_s^m,Z_s^m,U_s^n))\,\mathbb{1}_{\{\Phi(s)>N\}}ds, \\ \acute{J}_{2,t} &:= \beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}(f_n(s,Y_s^n,Z_s^n,U_s^n) - f(s,Y_s^n,Z_s^n,U_s^n))\,\mathbb{1}_{\{\Phi(s)\leq N\}}ds, \\ \acute{J}_{3,t} &:= \beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}(f(s,Y_s^n,Z_s^n,U_s^n) - f(s,Y_s^m,Z_s^m,U_s^n))\,\mathbb{1}_{\{\Phi(s)\leq N\}}ds, \\ \acute{J}_{4,t} &:= \beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}(f(s,Y_s^m,Z_s^m,U_s^n) - f_m(s,Y_s^m,Z_s^m,U_s^n))\,\mathbb{1}_{\{\Phi(s)\leq N\}}ds, \\ \acute{J}_{5,t} &:= \beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||f_m(s,Y_s^m,Z_s^m,U_s^n) - f_m(s,Y_s^m,Z_s^m,U_s^m)|ds, \end{aligned}$$

with the shorthand $\Phi(s) = |Y_s^n| + |Y_s^m| + |Z_s^n| + |Z_s^m| + \|U_s^n\|_\nu + \|U_s^m\|_\nu$. By using the fact that $|\hat{Y}_s^{n,m}| \leq \varphi_s^{\frac{1}{2}}$ and $\Phi(s) > N$, a simple computation shows that $\acute{J}_{1,t} \leq J_{1,t}$ and $\acute{J}_{2,t} + \acute{J}_{4,t} \leq J_{2,t}$. Finally, the inequalities $\acute{J}_{3,t} \leq J_{3,t}$ and $\acute{J}_{5,t} \leq J_{4,t}$ can be directly derived from Assumption (**A.3**)-(iii) and the Lipschitz condition with respect to u. □

Lemma A2. *Under Assumption of Proposition 1, we have*

$$\begin{aligned} & -C_{N,1}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\ & -\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds + J_{3,t} \\ \leq \quad & -\beta\frac{(\beta-1)}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds. \end{aligned}$$

Proof. The expression involving the process $(\hat{Z}_s^{n,m})$ in Proposition 1

$$\begin{aligned} & -\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\ & -\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds + \beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds. \end{aligned}$$

We have $|\hat{Y}_s^{n,m}|^2 \le \varphi_s := |\hat{Y}_s^{n,m}|^2 + (A_N)^{-1}$, since $\beta\frac{(2-\beta)}{2} > 0$, therefore

$$\begin{aligned}
&-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds + \beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds \\
\le\ & -\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\varphi_s ds - \frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds \\
&+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds \\
&+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds \\
=\ & \int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{C_{N,1}}{2}\varphi_s - \beta\frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2 + \beta M_2|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds.
\end{aligned}$$

If we choose $C_{N,1} := \beta\frac{2M_2^2}{\beta-1}\ln(A_N)$, then

$$\begin{aligned}
&-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds + \beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds \\
\le\ & \beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N) - \frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2 + M_2|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds \\
\le\ & \beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N) - \frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2 + M_2\sqrt{\varphi_s}|\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds.
\end{aligned}$$

The final inequality is derived from the fact that $|\hat{Y}_s^{n,m}| \le \sqrt{\varphi_s}$. We utilize Young's inequality ($ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$) by selecting $a = A|y|$, $b = z$, and $\varepsilon = \frac{\beta-1}{2}$.

$$A|y||z| - \frac{1}{\beta-1}A^2|y|^2 - \frac{(\beta-1)}{2}|z|^2 \quad \le \quad -\frac{\beta-1}{4}|z|^2.$$

For $A := M_2\sqrt{\ln(A_N)}$, $y := \sqrt{\varphi_s}$ and $z := |\hat{Z}_s^{n,m}|$, therefore

$$\begin{aligned}
&-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds \\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds + \beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds \\
\le\ & -\beta\frac{(\beta-1)}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds.
\end{aligned} \tag{A7}$$

□

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