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# Public private partnerships contract under moral hazard and ambiguous information

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This paper studies optimal Public Private Partnerships contract between a public entity and a consortium, with the possibility for the public to stop the contract. The public ("she") pays a continuous rent to the consortium ("he"), while the latter gives a response characterized by his effort. Usually, the public cannot observe the effort done by the consortium, in addition, the law of the Agent's effort which affects the distribution of the total output is to be chosen over a set of probabilities, leading to a Principal-Agent problem with moral hazard and Knightian uncertainty characterized by $\kappa$-ignorance. Our problem is formulated as a Stackelberg leadership model between the public and the consortium. This problem is solved in three steps. The first one consists on characterizing the worst case, then we derive the Agent's best response. In the third step, we solve the Principal problem by maximizing the social value of the project minus the rent paid, which is a standard stochastic control problem. The public value function is characterized by the solution of an associated Hamilton–Jacobi–Bellman variational inequality. The public value function, the optimal effort and the optimal rent are computed numerically in a feedback form by using the Howard algorithm. We show numerically that each increase in the degree of Knightian uncertainty leads to increase the optimal effort, and decrease the value function.

*Keywords*: Moral hazard; Knightian uncertainty; public private partnership; stochastic control; optimal stopping; Hamilton–Jacobi–Bellman variational inequality; Howard algorithm.

AMS Subject Classification: 60G40, 91B40, 91B70, 93E40

## 1. Introduction

Public private partnership (PPP) is defined as a long-term contract between a private party and a public entity, for the construction and/or the management of an asset or public service, in which the consortium bears the risks and a great

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responsibility to manage the project. Typically, the consortium is making an effort to improve the social value of the project in exchange for rent paid by the public. The objective of PPP contracts is to ensure a better quality-price ratio in the use of public funds. A major problem of this type of contracts is the asymmetry of information between the two parties, both in the negotiation and monitoring phases of the project. In particular, the public can usually not observe the effort done by the consortium. It is a Principal-Agent's problem with moral hazard.

The first paper on Principal-Agent problems in continuous-time is the paper of Holmstrom and Milgrom [8]. They considered a Brownian setting in which the Agent's effort affects only the drift of the output process, and receives a lump sum payment at the end of the contract, that is a finite time horizon. In their setting, the Principal is risk-neutral and the Agent is risk averse with Constant Absolute Risk Aversion (CARA) utility function. Holmstrom and Milgrom [8] considered a Stackelberg leadership model between the Principal and the Agent, that can be solved in two steps. First, given a fixed contract, the Principal computes the best answer of the Agent. Then, the Principal solves her problem by taking into account the best answer of the Agent and designs the optimal contract. Such model is suitable for short contract periods.

To solve our problem related to PPP, it is more convenient to randomize the horizon of the contract and follow Sannikov [13], which generalized the HM's model into a random time horizon in which the Principal pays continuous rent to the Agent, instead of paying at the end of the contract. He used an approach based on the dynamic programming principle to derive the Hamilton–Jacobi–Bellman (HJB) equation that must satisfies the principal value function, and thanks to a verification theorem, she derived the optimal contract. Such approach is powerful since the optimal rent and the optimal effort could be obtained in a feedback form and then could be approximated numerically by solving numerically the HJB equation. An alternative approach is used in the case of finite horizon by Williams [14] and in the monograph of Cvitanić and Zhang [2] and many other authors. It is based on the Pontryagin stochastic maximum principle, in models driven by Brownian Motion, to give necessary conditions for optimal efforts and contracts in terms of fully coupled system of forward backward stochastic differential equations. When the authors assume Markovian models, they can identify sufficient conditions via the standard approach of using HJB equations.

In Principal-Agent problems under moral hazard, many authors suppose that the Principal knows the probability distribution of the Agent's effort which is not obvious. In reality, the Principal may have some uncertainty or ambiguity on this probability, leading to several objective probability measures to consider. Preliminary results in the literature on uncertainty have been obtained in the case of dominated sets, namely with an objective reference probability measure (like drift uncertainty in Gilboa and Schmeidler [6]). Ambiguity or Knightian uncertainty has an important impact in economic problems. This notion was introduced by Knight [9], which has a major role in the economic contracts due to the inaccuracy of the available information. Risk and ambiguity are different. The first one refers to the situation where the probability distribution for each action is known and the second type which is more diffused, it consists in economic decision making where there is multitude probability distributions due to the inaccuracy of the information and cannot be reduced to a single probability distribution. Knight [9] showed this distinction, followed by Ellsberg [5] then Gilboa and Schmeidler [6]. They related the ambiguity with a concept of multiple priors in a atemporal version. Chen and Epstein [1] extended the version into an intertemporal multiple priors further, they introduced the notion "$\kappa$-ignorance" to characterize Knight uncertainty where $\kappa$ is the ignorance parameter. The situation is more ambiguous for the decision maker when the parameter $\kappa$ is larger. Dumav and Riedel [4] studied a moral hazard problem over a random horizon with continuous payments where the Principal and the Agent engage in a contractual relationship over unobservable effort that generates output with ambiguity. In contrast to Sannikov [13], they consider a model in which efforts map to sets of probability distributions. They characterized the optimal contract with ambiguous information. Mastrolia and Possamaï [11] studied the case when the Agent and the Principal faced uncertainty on the volatility of the output but in the finite maturity case.

In this paper, we consider a contract between a public entity and a consortium, in a continuous time setting. The consortium is making effort to improve the social value of the project, driven by a one-dimensional Brownian motion. The effort is not observable by the public and is ambiguous. The public must choose a continuous rent to pay the consortium in compensation to his effort. We assume that the effort only affects the drift and not the volatility of the social value. Indeed in a one-dimensional setting, controlling the volatility would imply that the effort is observable, through the quadratic variation of the social value. Our approach is inspired by the seminal paper of Sannikov [13]. In the first step, we prove the Agent value function under the worst possible scenario satisfies a backward stochastic differential equations (BSDE) with random horizon and we determine the Agent's best response, then we formulate the public value function as a standard stochastic control problem with the Agent value function as state variable and the contract and the Agent's best response as control processes. Thanks to the dynamic programming principle, we derive the Hamilton–Jacobi–Bellman variational inequality (HJBVI) characterizing the public value function. We approximated numerically the optimal rent and the optimal effort by using finite difference method and Howard algorithm. We obtained in a feedback form, the optimal effort and the optimal rent. We showed numerically that each increase in the degree of Knightian uncertainty leads to an increase of the effort and the decrease of the value function. Contrary to Dumav and Riedel [4], we assume that the diffusion of $X$ the social value of the project at time $t$ is not constant but depends of $X_t$, which adds a state variable in the HJBVI. We provide a rigorous mathematical framework using BSDE, stochastic control and optimal stopping technics. We detail the procedure to compute the numerical solutions.

The outline of the paper is as follows. In Sec. 2, we formulate the problem, using the weak approach and we describe the public and the consortium problems. In Sec. 3, we determine the incentive compatible contract under the worst case and we provide the dynamics of the consortium objective function, using the BSDE with random horizon technique. In Sec. 4, we derive the HJBVI associated to the public value function and we provide a verification theorem. Section 5 is devoted mainly to the numerical study of the HJBVI based on the Howard algorithm.

## 2. Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a one-dimensional Brownian motion $W := (W_t)_{t\geq 0}$ and we denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the completed natural filtration of $W$. The social value of the project $X_t$ is observed by both the Agent and the Principal, expressed as

$$X_t^x := x + \int_0^t \sigma(X_s^x) dW_s, \quad t \geq 0, \ \mathbb{P} \text{ a.s.}, \tag{2.1}$$

where

- $x > 0$ is the initial value of the project.
- $\sigma(\cdot)$ is the volatility of the operational cost of the infrastructure maintenance. The function $\sigma(\cdot)$ is Lipschitz and satisfies $\sigma_{\max} > \sigma(\cdot) > \sigma_{\min}$ where $\sigma_{\max}$ and $\sigma_{\min}$ are positive constants.

The Agent affects the project performance by taking an effort $A_t$, which changes the distribution of the process $W$. We define the martingale process $\gamma^A = (\gamma_t^A)_{t\geq 0}$ by

$$\gamma_t^A := \mathcal{E}\left(\int_0^t \frac{\varphi(A_s)}{\sigma(X_s^x)} dW_s\right), \quad t \geq 0, \ \mathbb{P} \text{ a.s.},$$

where $\mathcal{E}$ denotes the stochastic exponential and $\varphi$ is a function which will be specified hereafter.

Let $\mathbb{P}^A$ be a probability measure on $(\Omega, \mathcal{F})$, which is assumed to be equivalent to $\mathbb{P}$ and identified by its density $\gamma_t^A = \frac{d\mathbb{P}^A}{d\mathbb{P}}|_{\mathcal{F}_t}$, then by Girsanov's theorem

$$W_t^A = W_t - \int_0^t \frac{\varphi(A_s)}{\sigma(X_s^x)} ds \quad \text{for } t \geq 0$$

is $\mathbb{P}^A$-Brownian motion. Therefore, under $\mathbb{P}^A$ we have

$$X_t^x = x + \int_0^t \varphi(A_s) ds + \int_0^t \sigma(X_s^x) dW_s^A, \quad t \geq 0, \ \mathbb{P} \text{ a.s.} \tag{2.2}$$

In the Sannikov's formulation, when the effort $A$ is known, the probability measure $\mathbb{P}^A$ is fixed, meaning that the Agent knows the probability for describing the state process dynamics $(X_t)_{t\geq 0}$. In reality, the Agent may have some uncertainty on this probability, leading to several objective probability measures to consider. For this, we denote by $\theta$ $\mathbb{R}$-valued process that generates a probability measure $\mathbb{P}^\theta$ which is equivalent to $\mathbb{P}^A$. We denote by $\gamma^\theta$ the density of $\mathbb{P}^\theta$ with respect to $\mathbb{P}^A$ i.e. $\gamma^\theta_t = \frac{d\mathbb{P}^\theta}{d\mathbb{P}^A}|_{\mathcal{F}_t}$, which is defined by

$$\gamma^\theta_t := \mathcal{E}\left(\int_0^t \frac{\theta_s}{\sigma(X^x_s)} dW^A_s\right), \quad t \geq 0,\ \mathbb{P} \text{ a.s.},$$

where $\theta \in \Theta := \{(\theta_s)_{s\geq 0}\, \mathbb{F}$ progressively measurable process, $\theta_s \in [-\kappa, \kappa]\, ds \otimes d\mathbb{P}$ a.e.$\}$ and $\kappa$ is a positive constant. The corresponding set of different scenarios, called priors, is given by $\{\mathbb{P}^\theta, \theta \in \Theta\}$. By Girsanov theorem

$$W^\theta_t = W^A_t - \int_0^t \frac{\theta_s}{\sigma(X^x_s)} ds \quad \text{for } t \geq 0 \tag{2.3}$$

is $\mathbb{P}^\theta$ Brownian motion. Therefore, under $\mathbb{P}^\theta$, the social value of the project satisfies

$$X^x_t = x + \int_0^t (\varphi(A_s) + \theta_s) ds + \int_0^t \sigma(X^x_s) dW^\theta_s, \quad t \geq 0,\ \mathbb{P} \text{ a.s.} \tag{2.4}$$

We denote by $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times. We fix $\hat{p} \in (2, \infty)$ and we consider the set of admissible efforts

$$\mathcal{A}^{\hat{p}} := \Big\{(A_s)_{s\geq 0},\ \mathbb{F}\text{-progressively measurable processes},$$

$$A_s \geq 0,\ ds \otimes d\mathbb{P} \text{ a.e. and } \sup_{\tau \in \mathcal{T}} \sup_{\theta \in \Theta} \mathbb{E}^{\mathbb{P}}[(\gamma^{A,\theta}_\tau)^{\hat{p}}] < \infty\Big\}.$$

The public observes the social value $X$ of the project, but she cannot make the difference between $\int_0^\cdot \varphi(A_s)ds$ and $\int_0^\cdot \sigma(X^x_s)dW^A_s$, which implies that she does not observe directly the effort of the consortium: this is a situation of moral hazard. From equality (2.3), we deduce that $W^\theta$ is not observable by the Principal. She chooses the rent she will pay to the consortium to compensate him for his efforts and the operational costs that he supports. The public could end the contract at the date $\tau$, where $\tau$ is a stopping time in $\mathcal{T}$.

A contract is a triplet $\Gamma = ((R_t)_t, \tau, \xi)$ where $R$ is a non-negative $\mathbb{F}$-progressively measurable process, $\tau \in \mathcal{T}$ and $\xi$ is a non-negative $\mathcal{F}_\tau$-measurable random variable which represents the cost of stopping the contract.

We now define the respective optimization problems for the consortium and the public. Let us first define the functions involved in the formulation of the optimization problems.

**Assumption 2.1.**

- $\varphi$ is the function that models the marginal impact of the consortium's efforts on the social value, $\varphi : [0, \infty) \to [0, \infty)$ is $C^2$ strictly concave, increasing, the map $(x, a) \longrightarrow \frac{\varphi(a)}{\sigma(x)}$ is bounded, $\varphi(0) = 0$ and $\varphi'(0) > 0$. We denote by $\|\frac{\varphi}{\sigma}\|_\infty := \sup_{a\geq 0, x\in\mathbb{R}} \frac{|\varphi(a)|}{|\sigma(x)|}$.

- $U : [0, \infty) \to [0, \infty)$ is the utility function of the consortium, strictly concave increasing and satisfying $U(0) = 0$ and Inada's conditions $U'(\infty) = 0$, $U'(0) = \infty$.
- $h$ is the cost of the effort for the consortium; $h : [0, \infty) \to [0, \infty)$ is $C^2$, strictly convex increasing, $h(0) = 0$.
- The time preference parameter $\lambda$ of the consortium is greater than $\delta$ the one of the public ($\lambda \geq \delta$): the consortium is more impatient than the public.

We assume that given a contract $\Gamma = ((R_t)_t, \tau, \xi)$ offered by the Principal, the consortium gives a best response in terms of an effort process $A$: this is a Stackelberg leadership model. The latter is a game in which the principal is a leader that makes a proposition and the Agent, who is a follower, reacts sequentially. The consortium accepts the contract only if his expected payoff exceeds his reservation value $\underline{x}$.

The problem of the public and the consortium is solved in three steps. In the first step, we derive the worst scenario for the Agent $\theta^*$ given an effort $A$ and a contract $\Gamma$ i.e. $\theta^* = \theta^*(A, \Gamma)$. To alleviate notations, we omit $(A, \Gamma)$. Then we determine the best response of the Agent given $(\theta^*(A, \Gamma), \Gamma)$. The solution is denoted by $A^*(\theta^*(A, \Gamma), \Gamma)$. To alleviate notations, we omit $(\theta^*(A, \Gamma), \Gamma)$.

(1) In the first step and given $(A, R, \tau, \xi)$, we determine the worst possible scenario by solving

$$\theta^* \in \arg\min_{\theta \in \Theta} \mathbb{E}^\theta \left[ \int_0^\tau e^{-\lambda s}(U(R_s) - h(A_s))ds + e^{-\lambda \tau} U(\xi) \right],$$

The *objective function* starting from time $t$ for the Agent is

$$J_t^{\mathrm{amb}}(\Gamma, A, \theta) := \mathbb{E}^\theta \left[ \int_t^\tau e^{-\lambda(s-t)}(U(R_s) - h(A_s))ds \right.$$
$$\left. + e^{-\lambda(\tau - t)} U(\xi) | \mathcal{F}_t \right] \quad \forall t \in [\![0, \tau[\![ \quad \mathbb{P} \text{ a.s.}$$

(2) In the second step, we determine the consortium's best response under the worst possible scenario by solving

$$A^* \in \arg\max_{A \in \mathcal{A}_\rho^C} \mathbb{E}^A \left[ \int_0^\tau \gamma_s^{\theta^*} e^{-\lambda s}(U(R_s) - h(A_s))ds + \gamma_\tau^{\theta^*} e^{-\lambda \tau} U(\xi) \right],$$

where, for some $\rho > 0$,

$$\mathcal{A}_{\rho - 2\lambda}^C := \left\{ (A_s)_{s \geq 0} \in \mathcal{A}^{\hat{p}}, \text{ s.t. } \mathbb{E}^{\mathbb{P}} \left[ \int_0^\infty e^{(\rho - 2\lambda)s} |h(A_s)|^2 ds \right] < \infty \right.$$
$$\left. \text{and } \mathbb{E}^{\mathbb{P}} \left[ \int_0^\infty e^{(\rho - 2\lambda)s} |\varphi(A_s)|^2 ds \right] < \infty \right\}.$$

The *objective function* under the worst possible scenario starting from time $t$ for the Agent is

$$J_t^C(\Gamma, A) := \frac{1}{\gamma_t^{\theta^*}} \mathbb{E}^A \left[ \int_t^\tau \gamma_s^{\theta^*} e^{-\lambda(s-t)} (U(R_s) - h(A_s)) ds \right.$$
$$\left. + \gamma_\tau^{\theta^*} e^{-\lambda(\tau-t)} U(\xi) | \mathcal{F}_t \right] \quad \forall t \in [\![0, \tau[\![ \quad \mathbb{P} \text{ a.s.}$$

(3) Given the best response of the consortium and under the worst case, the public problem is formulated by

$$\sup_{\Gamma \in \mathcal{A}_\rho^P} \sup_{\mathbb{P}^{A^*} \in \mathcal{P}} \mathbb{E}^{A^*} \left[ \int_0^\tau \gamma_s^{\theta^*} e^{-\delta s} (\varphi(A_s^*) + \theta_s^* - R_s) ds - \gamma_\tau^{\theta^*} e^{-\delta \tau} \xi \right], \tag{2.5}$$

subject to the reservation constraint

$$\mathbb{E}^{A^*} \left[ \int_0^\tau \gamma_s^{\theta^*} e^{-\lambda s} (U(R_s) - h(A_s^*)) ds + \gamma_\tau^{\theta^*} e^{-\lambda \tau} U(\xi) \right] \geq \underline{x},$$

where

$$\mathcal{A}_{\rho-2\lambda}^P := \Big\{ ((R_s)_{s \geq 0}, \tau, \xi) \text{ s.t. } (R_s)_{s \geq 0}$$
$$\text{is a } \mathbb{F}\text{-progressively measurable process}, R_s \geq 0 \ ds \otimes d\mathbb{P} \text{ a.e.},$$
$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^\tau e^{(\rho - 2\lambda)s} (U(R_s)^2 \vee R_s^2) ds \right] < \infty, \ \tau \in \mathcal{T},$$
$$\xi \text{ non-negative } \mathcal{F}_\tau\text{-measurable, and}$$
$$\mathbb{E}^{\mathbb{P}} [e^{(\rho - 2\lambda)\tau} (|U(\xi)|^2 \vee \xi^2) \mathbf{1}_{\{\tau < +\infty\}}] < \infty \Big\}$$

and

$$\mathcal{P} = \{ \mathbb{P}^{A^*} \sim \mathbb{P}, \ A^* \in \mathcal{A}_\rho^C \}.$$

The *objective function* starting from time $t$ for the Principal is

$$J_t^P(\Gamma, A^*) := \frac{1}{\gamma_t^{\theta^*}} \mathbb{E}^{A^*} \left[ \int_t^\tau \gamma_s^{\theta^*} e^{-\delta(s-t)} (\varphi(A_s^*) + \theta_s^* - R_s) ds \right.$$
$$\left. - \gamma_\tau^{\theta^*} e^{-\delta(\tau-t)} \xi | \mathcal{F}_t \right] \quad \forall t \in [\![0, \tau[\![ \quad \mathbb{P} \text{ a.s.}$$

## 3. Incentive Compatible Contracts

In this section, we will determine the incentive compatible contracts and provide the dynamics of the consortium objective function $J^C$. We will prove that $J^C$ is the unique solution for a certain type of a BSDE with random horizon of the following form:

$$Y_t = \zeta \mathbf{1}_{\{\tau<+\infty\}} + \int_t^\tau g(s,\omega,X_s,Y_s,Z_s)ds - \int_t^\tau Z_s dW_s, \tag{3.1}$$

where the generator $g : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

BSDEs with random horizon have been studied by some authors. Darling and Pardoux [3] studied a BSDE with random horizon. Here, we assume that:

(H1) The generator $g : [0,\infty) \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies, for each $y, z \in \mathbb{R}$, $g(\cdot,\cdot,y,z)$ is a progressively measurable process and for some $\eta > 0$, we have

$$\mathbb{E}\left[e^{\eta\tau}|\zeta|^2\mathbf{1}_{\{\tau<+\infty\}} + \int_0^\tau e^{\eta s}|g(s,\omega,X_s,0,0)|^2 ds\right] < \infty.$$

(H2) $g$ is Lipschitz with respect to $y$ and $z$, i.e. there exists a constant $C_g > 0$ such that for any $s$, $\omega$, $x$,

$$|g(s,\omega,x,y_1,z_1) - g(s,\omega,x,y_2,z_2)| \le C_g(|y_1-y_2| + |z_1-z_2|)ds \otimes d\mathbb{P} \text{ a.e.}$$

We introduce the following spaces for a fixed stopping time $\tau \in \mathcal{T}$ and for some $\eta > 0$:

- $\mathcal{M}_\eta(0,\tau;\mathbb{R})$ the set of $\mathbb{F}$-progressively measurable, $\mathbb{R}$-valued processes on $\Omega \times [\![0,\tau]\!]$.
- $\mathcal{H}^2_\eta(0,\tau;\mathbb{R}) = \{Z \in \mathcal{M}_\eta(0,\tau;\mathbb{R}) \text{ s.t. } \mathbb{E}[\int_0^\tau e^{\eta t}|Z_t|^2 dt] < +\infty\}$.
- $\mathcal{S}^2_\eta(0,\tau;\mathbb{R}) = \{Y \in \mathcal{M}_\eta(0,\tau;\mathbb{R}) \text{ s.t. } \mathbb{E}[\sup_{0\le t\le\tau} e^{\eta t}|Y_t|^2] < +\infty\}$.

The existence and uniqueness of a solution to the BSDE (3.1) are given in the following theorem proved in [3] (see Theorem 3.4 and Corollary 4.4.2).

**Theorem 3.1.** *Let* (H1)−(H2) *be fulfilled, then there exists a unique solution* $(Y,Z)$ *of the BSDE* (3.1) *in* $\mathcal{H}^2_{\rho-2\lambda}(0,\tau;\mathbb{R}) \times \mathcal{H}^2_{\rho-2\lambda}(0,\tau;\mathbb{R})$, *moreover* $Y \in \mathcal{S}^2_{\rho-2\lambda}(0,\tau;\mathbb{R})$, *i.e.* $\mathbb{E}[\sup_{0\le t\le\tau} e^{(\rho-2\lambda)t}|Y_t|^2] < +\infty$. *Let* $(Y^1,Z^1)$ *and* $(Y^2,Z^2)$ *be two solutions of the BSDEs associated with parameters* $(g^1,\xi,\tau)$ *and* $(g^2,\xi,\tau)$. *We suppose that* $g^1(t,\omega,X_t,Y^1,Z^1) \le g^2(t,\omega,X_t,Y^1,Z^1)dt \otimes d\mathbb{P}$ *a.e., then* $Y^1_t \le Y^2_t$, $\forall t \in [\![0,\tau[\![ \quad \mathbb{P}$ *a.s.*

Theorem 3.1 is used to determine the worst case for the Agent, the incentive compatible contract and to provide the dynamics of the consortium objective function. The key tool to solve the two first steps is the comparison theorem.

**Proposition 3.1.** *Let* $\Gamma \in \mathcal{A}^p_{\rho-2\lambda}$, $A \in \mathcal{A}^C_{\rho-2\lambda}$ *and* $\theta \in \Theta$, *there exists* $Z^{A,\theta} \in \mathcal{H}^2_{\rho-2\lambda}(0,\tau;\mathbb{R})$ *such that the dynamics of the Agent objective function evolves* *according to the BSDE with random horizon*

$$\begin{cases} -dJ_t^{\mathrm{amb}}(\Gamma, A, \theta) = \Big(-\lambda J_t^{\mathrm{amb}}(\Gamma, A, \theta) + U(R_t) + \psi(A_t, X_t^x, Z_t^{A,\theta}) \\ \qquad\qquad + \dfrac{\theta_t}{\sigma(X_t^x)} Z_t^{A,\theta}\Big)\, dt - Z_t^{A,\theta} dW_t, \\ J_\tau^{\mathrm{amb}}(\Gamma, A, \theta) = U(\xi)\mathbf{1}_{\{\tau<+\infty\}} \end{cases} \tag{3.2}$$

*and*

$$J_t^{\mathrm{amb}}(\Gamma, A, \theta) \geq J_t^C(\Gamma, A) := J_t^{\mathrm{amb}}(\Gamma, A, \theta^*(Z^{A,\theta})) \quad \forall\, t \in [\![0, \tau[\![ \quad \mathbb{P}\ a.s., \tag{3.3}$$

*where*

$$\begin{cases} \psi(A_t, X_t^x, Z_t^{A,\theta}) := -h(A_t) + \dfrac{\varphi(A_t)}{\sigma(X_t^x)} Z_t^{A,\theta}, \\ \theta^*(Z_t^{A,\theta}) := \arg \min\limits_{\alpha\in[-\kappa,\kappa]} \left(\alpha Z_t^{A,\theta}\right) = -\kappa\, \mathrm{sgn}(Z_t^{A,\theta}). \end{cases}$$

**Proof.** For an admissible contract $\Gamma \in \mathcal{A}^P_{\rho-2\lambda}$, an effort $A \in \mathcal{A}^C_{\rho-2\lambda}$, an ambiguity process $\theta \in \Theta$, and for any $t \in [\![0, \tau[\![$, we define the process

$$M_t(\Gamma, A, \theta) := \tilde{J}_t^{\mathrm{amb}}(\Gamma, A, \theta) + \int_0^t (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds,$$

where $\tilde{J}^{\mathrm{amb}}$ is the discounted Agent objective function i.e. $\tilde{J}_t^{\mathrm{amb}}(\Gamma, A, \theta) = e^{-\lambda t} J_t^{\mathrm{amb}}(\Gamma, A, \theta)\, dt \otimes dP$ and $\tilde{U}$ is the discounted utility i.e. $\tilde{U}(R_t) := e^{-\lambda t} U(R_t)\, dt \otimes dP$. From the definition of $J_t^{\mathrm{amb}}$ and by Bayes formula under $\mathbb{P}^A$ and $\mathbb{P}^\theta$, we deduce

$$\begin{aligned} &M_t(\Gamma, A, \theta) \\ &\quad = \mathbb{E}^\theta \left[ \int_0^\tau (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds + e^{-\lambda\tau} \xi \mathbf{1}_{\{\tau<+\infty\}} | \mathcal{F}_t \right] \\ &\quad = \frac{1}{\gamma_t^\theta} \mathbb{E}^{\mathbb{A}} \left[ \gamma_\tau^\theta \left( \int_0^\tau (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds + e^{-\lambda\tau} \xi \mathbf{1}_{\{\tau<+\infty\}} \right) \Big| \mathcal{F}_t \right] \\ &\quad = \frac{1}{\gamma_t^\theta \gamma_t^A} \mathbb{E}^{\mathbb{P}} \left[ \gamma_\tau^\theta \gamma_\tau^A \left( \int_0^\tau (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds + e^{-\lambda\tau} \xi \mathbf{1}_{\{\tau<+\infty\}} \right) \Big| \mathcal{F}_t \right]. \end{aligned}$$

A straightforward calculus shows that

$$\gamma_t^{A,\theta} := \gamma_t^\theta \gamma_t^A := \mathcal{E}\left( \int_0^\cdot \left( \frac{\varphi(A_s) + \theta_s}{\sigma(X_s^x)} \right) dW_s \right)_t.$$

This shows that the process $\gamma_t^{A,\theta} M_t(\Gamma, A, \theta)$ is $(\mathbb{P}, \mathbb{F})$-local-martingale. As a result of the martingale representation theorem there exists a unique progressively measurable process $\chi$ such that

$$d\gamma_t^{A,\theta} M_t(\Gamma, A, \theta) = \chi_t dW_t.$$

Applying Itô's formula to $\gamma_t^{A,\theta} M_t(\Gamma, A, \theta)$, we obtain

$$
\begin{aligned}
&dM_t(\Gamma, A, \theta) \\
&\quad = \frac{1}{\gamma_t^{A,\theta}} (d(\gamma_t^{A,\theta} M_t(\Gamma, A, \theta)) - M_t(\Gamma, A, \theta) d\gamma_t^{A,\theta} - d\langle M(\Gamma, A, \theta), \gamma^{A,\theta} \rangle_t) \\
&\quad = \frac{1}{\gamma_t^{A,\theta}} \left( \left( \chi_t - M_t(\Gamma, A, \theta) \gamma_t^{A,\theta} \frac{\varphi(A_t) + \theta_t}{\sigma(X_t^x)} \right) dW_t - d \left\langle M(\Gamma, A, \theta), \gamma^{A,\theta} \right\rangle_t \right) \\
&\quad = \left( \frac{\chi_t}{\gamma_t^{A,\theta}} - M_t(\Gamma, A, \theta) \frac{\varphi(A_t) + \theta_t}{\sigma(X_t^x)} \right) dW_t - \frac{1}{\gamma_t^{A,\theta}} d \left\langle M(\Gamma, A, \theta), \gamma^{A,\theta} \right\rangle_t .
\end{aligned} \tag{3.4}
$$

We define the process $\tilde{Z}_{\cdot}^{A,\theta}$ by

$$
\tilde{Z}_t^{A,\theta} := \frac{\chi_t}{\gamma_t^{A,\theta}} - M_t(\Gamma, A, \theta) \frac{\varphi(A_t) + \theta_t}{\sigma(X_t^x)} \, dt \otimes dP \text{ a.e.}, \tag{3.5}
$$

then the quadratic variation of $M(\Gamma, A, \theta)$ and $\gamma^{A,\theta}$ satisfies

$$
d\langle M(\Gamma, A, \theta), \gamma^{A,\theta} \rangle_t = \tilde{Z}_t^{A,\theta} \gamma_t^{A,\theta} \frac{\varphi(A_t) + \theta_t}{\sigma(X_t^x)} dt.
$$

Using Eqs. (3.4) and (3.5), we deduce that

$$
dM_t(\Gamma, A, \theta) = -\tilde{Z}_t^{A,\theta} \frac{\varphi(A_t) + \theta_t}{\sigma(X_t^x)} dt + \tilde{Z}_t^{A,\theta} dW_t.
$$

From the definition of $\tilde{J}^{\text{amb}}$, we obtain

$$
\begin{cases}
-d\tilde{J}_t^{\text{amb}}(\Gamma, A, \theta) = \left( \tilde{U}(R_t) + \tilde{\psi}(A_t, X_t^x, \tilde{Z}_t^{A,\theta}) + \tilde{Z}_t^{A,\theta} \dfrac{\theta_t}{\sigma(X_t^x)} \right) dt - \tilde{Z}_t^{A,\theta} dW_t, \\
\tilde{J}_\tau^{\text{amb}}(\Gamma, A, \theta) = e^{-\lambda\tau} U(\xi) \mathbf{1}_{\{\tau < +\infty\}},
\end{cases}
$$

where $\tilde{\psi}(A_t, X_t^x, \tilde{Z}_t^{A,\theta}) := -e^{-\lambda t} h(A_t) + \frac{\varphi(A_t)}{\sigma(X_t^x)} \tilde{Z}_t^{A,\theta}$ $dt \otimes dP$ a.e. By Itô's formula, we show that $(J^{\text{amb}}, Z^{A,\theta})$ satisfies the BSDE (3.2) whose generator is defined by $g(t, \omega, X_t^x, y, z) = -\lambda y + U(R_t) + \psi(A_t, X_t^x, z) + z \frac{\theta_t}{\sigma(X_t^x)}$. Under the integrability assumptions on $A$ and $\Gamma$, assumption (H1) is satisfied. As the function $\frac{\varphi}{\sigma}$ is bounded (see Assumption 2.1), the process $\theta$ is bounded and $\sigma$ is supposed to be bounded from below, then there exists a positive constant $K$ such that

$$
\left| \frac{\varphi(A_t) + \theta_t}{\sigma(X_t^x)} \right| \leq K \; dt \otimes d\mathbb{P} \text{ a.e.}, \tag{3.6}
$$

thus the generator $g$ of the BSDE (3.2) is uniformly Lipschitz with respect to $(y, z)$ and therefore satisfies assumption (H2). Then, there exists a unique solution

$(Y, Z^{A,\theta}) \in \mathcal{S}^2_{\rho-2\lambda}(0,\tau;\mathbb{R}) \times \mathcal{H}^2_{\rho-2\lambda}(0,\tau;\mathbb{R})$ of the BSDE (3.2). Since for any $\theta \in \Theta$,

$$Z_t^{A,\theta}\theta_t \geq -|Z_t^{A,\theta}|\kappa = Z_t^{A,\theta}\theta^*(Z_t^{A,\theta})\ dt \otimes d\mathbb{P}\ \text{ a.e.}$$

By Theorem 3.1, we have

$$J_t^{\mathrm{amb}}(\Gamma, A, \theta) \geq J_t^{\mathrm{amb}}(\Gamma, A, \theta^*(Z_t^{A,\theta})) \quad \forall\, t \in [\![0,\tau[\![ \quad \mathbb{P}\text{ a.s.} \tag{3.7}$$

and so inequality (3.3) is proved. □

In the following proposition, we derive the BSDE that satisfies the Agent objective function under the worst case.

**Proposition 3.2.** *Let $\Gamma \in \mathcal{A}^P_{\rho-2\lambda}$, $A \in \mathcal{A}^C_{\rho-2\lambda}$. Then, the BSDE that satisfies $J^C(\Gamma, A)$, is given by*

$$\begin{cases} -dJ_t^C(\Gamma, A) = \left(-\lambda J_t^C(\Gamma, A) + U(R_t) + \psi(A_t, X_t^x, Z_t^A) - |Z_t^A|\dfrac{\kappa}{\sigma(X_t^x)}\right) dt \\ \qquad\qquad\qquad -\ Z_t^A dW_t, \\ J_\tau^C(\Gamma, A) = U(\xi)\mathbf{1}_{\{\tau<+\infty\}}, \end{cases}$$

*where $\psi$ is defined in Lemma* 3.1.

**Proof.** For $\Gamma \in \mathcal{A}^P_{\rho-2\lambda}$ and $A \in \mathcal{A}^C_{\rho-2\lambda}$, we consider the BSDE

$$\begin{cases} -dY_t = \left(-\lambda Y_t + U(R_t) + \psi(A_t, X_t^x, Z_t^A) - |Z_t^A|\dfrac{\kappa}{\sigma(X_t^x)}\right) dt - Z_t^A dW_t, \\ Y_\tau = U(\xi)\mathbf{1}_{\{\tau<+\infty\}}, \end{cases} \tag{3.8}$$

assumptions (H1) and (H2) are satisfied. From Theorem (3.1), there exists a unique pair $(Y, Z^A) \in \mathcal{S}^2_{\rho-2\lambda}(0,\tau;\mathbb{R}) \times \mathcal{H}^2_{\rho-2\lambda}(0,\tau;\mathbb{R})$ that solves BSDE (3.8). On the other hand, $J^{\mathrm{amb}}(\Gamma, A, \theta^*)$ solves

$$\begin{aligned} -dJ_t^{\mathrm{amb}}(\Gamma, A, \theta^*) &= \Big(-\lambda J_t^{\mathrm{amb}}(\Gamma, A, \theta^*) + U(R_t) + \psi(A_t, X_t^x, Z_t^{A,\theta^*}) \\ &\qquad + \frac{\theta_t^*}{\sigma(X_t^x)} Z_t^{A,\theta^*}\Big) dt - Z_t^{A,\theta^*} dW_t \\ &= \Big(-\lambda J_t^{\mathrm{amb}}(\Gamma, A, \theta^*) + U(R_t) + \psi(A_t, X_t^x, Z_t^{A,\theta^*}) \\ &\qquad - |Z_t^{A,\theta^*}|\frac{\kappa}{\sigma(X_t^x)}\Big) dt - Z_t^{A,\theta^*} dW_t, \end{aligned}$$

where the terminal condition is given by $J_\tau^{\mathrm{amb}}(\Gamma, A, \theta^*) = U(\xi)\mathbf{1}_{\{\tau<+\infty\}}$. The unicity of the solution of the BSDE (3.8) implies that $(J^C(\Gamma, A) = J^{\mathrm{amb}}(\Gamma, A, \theta^*), Z^A)$ solves (3.8) and so the proposition is proved. □

**Remark 3.1.** Hajjej *et al.* [7] proved that for any admissible contract $\Gamma \in \mathcal{A}^P_{\rho-2\lambda}$, the best response of the consortium $A^* \in \mathcal{A}^C_{\rho-2\lambda}$ could be written as a deterministic function which depends on the social value of the project and the process $Z$ i.e.

$$A^*_t = A^*(X^x_t, Z^{A,\theta^*}_t) = \left(\frac{h'}{\varphi'}\right)^{-1}\left(\frac{Z^{A,\theta^*}_t}{\sigma(X^x_t)}\right)\mathbf{1}_{\{Z^{A,\theta^*}_t>0\}}. \tag{3.9}$$

The following proposition gives the dynamics of $J^C$ for any incentive compatible contract.

**Proposition 3.3.** *We assume that the generator defined by* $(t,\omega,x,y,z) \mapsto g(t,\omega,x,y,z) := -\lambda y + U(R_t) - h(A^*(z)) + z\frac{\varphi(A^*(z))}{\sigma(x)} - |z|\frac{\kappa}{\sigma(x)}$ *satisfies the assumptions* (H1) *and* (H2) *with Lipschitz coefficient* $C_g$ *satisfying* $C_g^2 < \rho$. *Then, the dynamics of* $J^C$ *for any incentive compatible contract* $(\Gamma, A^*(X^x, Z))$ *is given by the BSDE with random terminal condition*

$$\begin{cases} dJ^C_t(\Gamma, A^*(X^x,Z)) = -\Big(-\lambda J^C_t(\Gamma, A^*(X^x,Z)) + U(R_t) \\ \qquad\qquad + \psi(A^*(X^x_t,Z_t), X^x_t, Z_t) - |Z_t|\dfrac{\kappa}{\sigma(X^x_t)}\Big)\,dt + Z_t dW_t, \\ J^C_\tau(\Gamma, A^*(X^x,Z)) = U(\xi)\mathbf{1}_{\{\tau<+\infty\}}, \end{cases}$$

*where* $A^*(X^x,Z)$ *is defined by* (3.9).

**Proof.** For $\Gamma \in \mathcal{A}^P_{\rho-2\lambda}$ and $A^*(X^x,Z) \in \mathcal{A}^C_{\rho-2\lambda}$, we consider the BSDE

$$\begin{cases} dY_t = -\Big(-\lambda Y_t + U(R_t) + \psi(A^*(X^x_t,Z_t), X^x_t, Z_t) - |Z_t|\dfrac{\kappa}{\sigma(X^x_t)}\Big)\,dt + Z_t dW_t, \\ Y_\tau = U(\xi)\mathbf{1}_{\{\tau<+\infty\}}. \end{cases} \tag{3.10}$$

From the assumption of the proposition, (H1) and (H2) are satisfied. From Theorem 3.1, there exists a unique solution $(Y,Z) \in \mathcal{S}^2_{\rho-2\lambda}(0,\tau;\mathbb{R}) \times \mathcal{H}^2_{\rho-2\lambda}(0,\tau;\mathbb{R})$ to the BSDE (3.10). On the other hand, applying the martingale representation theorem to $(J^C_t(\Gamma, A^*(X^x,Z)))_{t\geq 0}$ as in Lemma 3.1, there exists a progressively measurable process $(Z^{A^*(X^x,Z)}_t)_t$ such that

$$\begin{cases} dJ^C_t(\Gamma, A^*(X^x,Z)) \\ \quad = -\Big(-\lambda J^C_t(\Gamma, A^*(X^x,Z)) + U(R_t) + \psi(A^*(X^x_t,Z_t), X^x_t, Z^{A^*(X^x,Z)}_t) \\ \qquad - |Z^{A^*(X^x,Z)}_t|\dfrac{\kappa}{\sigma(X^x_t)}\Big)dt + Z^{A^*(X^x,Z)}_t dW_t, \\ J^C_\tau(\Gamma, A^*(X^x,Z)) = U(\xi)\mathbf{1}_{\{\tau<+\infty\}}. \end{cases}$$

The uniqueness of the solution yields that

$$Z_t = Z_t^{A^*(X^x,Z)} \quad \forall t \in [\![0,\tau[\![ \ \mathbb{P} \text{ a.s.}$$

Moreover, as $\psi(A_t, X_t^x, Z_t) \le \psi(A^*(X_t^x, Z_t), X_t^x, Z_t)$ $\forall t \in [\![0,\tau[\![$ $\mathbb{P}$ a.s. $\forall A \in \mathcal{A}_\rho^C$, by the comparison theorem (see Theorem 3.1), we have for all $A \in \mathcal{A}_{\rho-2\lambda}^C$,

$$J_t^C(\Gamma, A) \le J_t^C(\Gamma, A^*(X^x, Z)), \quad \forall t \in [\![0,\tau[\![ \quad \mathbb{P} \text{ a.s.} \qquad \square$$

**Remark 3.2.** Hajjej *et al.* [7] proved that $(J_t^C(\Gamma, A^*(X^x, Z)))_{t\ge 0}$ solves the BSDE

$$\begin{cases} dJ_t^C(\Gamma, A^*(X^x, Z)) = -(-\lambda J_t^C(\Gamma, A^*(X^x, Z)) + U(R_t) \\ \qquad\qquad\qquad\qquad + \psi(A^*(X_t^x, Z_t), X_t^x, Z_t))dt + Z_t dW_t, \\ J_\tau^C(\Gamma, A^*(X^x, Z)) = U(\xi)\mathbf{1}_{\{\tau<+\infty\}}, \end{cases}$$

which is different from the BSDE given in Proposition 3.3. This shows that the optimal effort $A^*(X^x, Z)$, given by (3.9), with and without ambiguity are not identical since the processes $Z$ with and without ambiguity are not identical.

**Example 3.1.** We fix $\alpha > 0$ and $\beta > 0$. We define $\varphi$ and $h$ as follows: $\varphi(x) := 1 - \exp(-\alpha x)$ and $h(x) := \exp(\beta x) - 1$. A straightforward calculus shows that

$$A^*(x,z) = \begin{cases} \dfrac{1}{\alpha+\beta} \log\left(\dfrac{\alpha z}{\beta\sigma(x)}\right) & \text{if } z > \sigma(x)\dfrac{\beta}{\alpha}, \\ 0 & \text{if } z \le \sigma(x)\dfrac{\beta}{\alpha}. \end{cases}$$

Then, the generator of the BSDE (3.10) is given by

$$g(t,\omega,x,y,z) = \begin{cases} -\lambda y + U(R_t) + 1 - \left(\dfrac{\alpha z}{\beta\sigma(x)}\right)^{\frac{\beta}{\alpha+\beta}} + \dfrac{z}{\sigma(x)} - \left(\dfrac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\left(\dfrac{z}{\sigma(x)}\right)^{\frac{\beta}{\alpha+\beta}} \\ \quad - \dfrac{\kappa}{\sigma(x)}|z| \quad \text{if } z > \sigma(x)\dfrac{\beta}{\alpha}, \\ -\lambda y + U(R_t) - \dfrac{\kappa}{\sigma(x)}|z| \quad \text{if } z \le \sigma(x)\dfrac{\beta}{\alpha}. \end{cases}$$

The generator $g$ satisfies the Lipschitz property with respect to $(y,z)$ and

$$\mathbb{E}\left[\int_0^\tau e^{(\rho-2\lambda)s}|g(s,\omega,X_s,0,0)|^2 ds\right] = \mathbb{E}\left[\int_0^\tau e^{(\rho-2\lambda)s}|U(R_s)|^2 ds\right] < \infty.$$

It yields that the generator $g$ satisfies assumptions (H1) and (H2).

## 4. Solving the Principal Problem

The previous section gives a complete characterization of the incentive compatible contracts. The objective consortium process for any incentive compatible contracts satisfies a well-posed BSDE with random horizon. The public wants to offer contracts which reveal the action of the consortium i.e. incentive compatible contracts. As a consequence, the stochastic control problem of the Principal, reformulated as a stochastic control where $J^C$ is a state variable and the contract $\Gamma$ and the best effort $A^*(X, Z)$ are control processes, is a standard one. Her value function is given by

$$v(x,y) = \sup_{(R,\tau,A^*(X^x,Z))\in\mathcal{Y}} \mathbb{E}^{A^*(X^x,Z)}\Bigg[\int_0^\tau e^{-\delta s}(\varphi(A^*(X_s^x,Z_s)) \\ -\kappa\mathbf{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds - e^{-\delta\tau}U^{-1}(J_\tau^{C,y})\Bigg], \tag{4.1}$$

where $y$ is the initial value of $(J_t^{C,y})_{t\geq 0}$, which evolves according to the following stochastic differential equations (SDE):

$$\begin{cases} dJ_t^{C,y} = \left(\lambda J_t^{C,y} - U(R_t) - \psi(A^*(X_t^x,Z_t),X_t^x,Z_t) + |Z_t|\dfrac{\kappa}{\sigma(X_t^x)}\right)dt + Z_t dW_t, \\ J_0^{C,y} = y \geq \underline{x}, \end{cases} \tag{4.2}$$

and the set $\mathcal{Y}$ is defined as

$$\mathcal{Y} := \Bigg\{(R,\tau,A^*(X^x,Z)) \text{ s.t. } (R_s)_{s\geq 0} \\ \text{is } \mathbb{F}\text{-progressively measurable non-negative process,} \\ \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau e^{(\rho-2\lambda)s}(U(R_s)^2 \vee R_s^2)ds\right] < \infty,\ \tau\in\mathcal{T},\ A^*(X^x,Z)\in\mathcal{A}_{\rho-2\lambda}^C\Bigg\}.$$

Thanks to the condition $J_0^{c,y} = y \geq \underline{x}$, the reservation constraint of the Agent is satisfied, since the initial value required by the Agent is at least $\underline{x}$. We give the dynamic programming principle which is a fundamental principle in the theory of stochastic control. It is formulated as follows: For any stopping time $\zeta\in\mathcal{T}$,

$$v(x,y) = \sup_{(R,\tau,A^*(X^x,Z))\in\mathcal{Y}} \mathbb{E}^{A^*(X^x,Z)}\Bigg[\int_0^{\tau\wedge\zeta} e^{-\delta s}(\varphi(A^*(X_s^x,Z_s)) - \kappa\mathbf{1}_{\{A^*(X_s^x,Z_s)>0\}} \\ - R_s)ds - e^{-\delta\tau}U^{-1}(J_\tau^{C,y})\mathbf{1}_{\tau<\zeta} + e^{-\delta\zeta}v(X_\zeta^x, J_\zeta^{C,y})\mathbf{1}_{\zeta\leq\tau}\Bigg]. \tag{4.3}$$

The HJBVI satisfied by the public value function is the infinitesimal version of the dynamic programming principle. It describes the local behavior of the value function by sending the stopping time $\zeta$ in (4.3) to the initial time. The HJBVI defined on $\mathbb{R} \times (0, \infty)$ is given by

$$\min\left\{\delta w(x,y) - \sup_{(r,a)\in\mathbb{R}_+\times\mathbb{R}_+}[\mathcal{L}^{a,r}w(x,y) + \varphi(a) - \kappa\mathbf{1}_{\{a>0\}} - r], w(x,y) + U^{-1}(y)\right\} = 0, \tag{4.4}$$

where $\mathcal{L}^{a,r}$ is the generator associated with the SDE (4.2). It is given by

$$\begin{aligned}\mathcal{L}^{a,r}w(x,y) := {} & \frac{1}{2}\sigma^2(x)\frac{\partial w^2(x,y)}{\partial x^2} + \frac{1}{2}\left(\sigma(x)\frac{h'(a)}{\varphi'(a)}\right)^2 \mathbb{1}_{\{a>0\}}\frac{\partial w^2(x,y)}{\partial y^2} \\ & + \sigma^2(x)\frac{h'(a)}{\varphi'(a)}\mathbb{1}_{\{a>0\}}\frac{\partial w^2(x,y)}{\partial x \partial y} + (\varphi(a) - \kappa\mathbb{1}_{\{a>0\}})\frac{\partial w(x,y)}{\partial x} \\ & + [\lambda y - U(r) + h(a)]\frac{\partial w(x,y)}{\partial y}.\end{aligned}$$

In the following lemma, we give the boundary condition when $y = 0$.

**Lemma 4.1.** *The value function $v$ satisfies the boundary conditions*:

$$v(x,0) = 0 \quad \textit{for all } x \in \mathbb{R}. \tag{4.5}$$

**Proof.** We fix $x \in \mathbb{R}$. We have $J_0^{C,0} = 0$. As the Agent can guarantee himself non-negative utility by taking effort 0, the consortium objective function is non-negative almost surely i.e. $J_t^{C,y} \geq 0$ for all $t \in [\![0, \tau^*[\![$ $\mathbb{P}$ a.s. Since $J_t^{C,y}$ satisfies the SDE (4.2) and for obtaining a non-negative solution, we must have $Z_t = 0$ for all $t \in [\![0, \tau^*[\![$ $\mathbb{P}$ a.s. From the bijection between $Z$ and $A^*(X, Z)$, we must have $A^*(X_t, Z_t) = 0$ for all $t \in [\![0, \tau^*[\![$ $\mathbb{P}$ a.s. Then from the definition of the value function, is optimal for the public to choose $R_t^* = 0$ for all $t \in [\![0, \tau^*[\![$ $\mathbb{P}$ a.s. As the drift of the SDE (4.2) is equal to 0, then it is optimal for the public to stop the contract at $\tau^* = 0$ $\mathbb{P}$ a.s. This shows $v(x, 0) = 0$ for all $x \in \mathbb{R}$. □

The next lemma gives an upper and a lower bound of the value function $v$.

**Lemma 4.2.** *Under Assumption* 2.1, *there exists a positive constant $L > 0$, s.t. for all $(x, y) \in \mathbb{R} \times \mathbb{R}^+$, we have*

$$|v(x,y)| \leq L + U^{-1}(y). \tag{4.6}$$

**Proof.** From the definition of the value function (see (4.1)), and for all $(R, 0, A^*(Z)) \in \mathcal{Y}$, we have

$$v(x,y) \geq -U^{-1}(y).$$

Since $J_\tau^{C,y}$ is non-negative $\mathbb{P}$ a.s., and the rent is non-negative, then

$$v(x,y) \leq \sup_{(R,\tau,A^*(X^x,Z))\in\mathcal{Y}} \mathbb{E}^{A^*(X^x,Z)}\left[\int_0^{\tau\wedge\zeta} \gamma_s^{\theta^*(Z)} e^{-\delta s}\varphi(A^*(X_s^x,Z_s)ds\right].$$

From Assumption 2.1 and since $\sigma(X_\cdot^x)$ is bounded, we have

$$\begin{aligned} v(x,y) &\leq \sigma_{\max}\left\|\frac{\varphi}{\sigma}\right\|_\infty \sup_{(R,\tau,A^*(X^x,Z))\in\mathcal{Y}} \mathbb{E}^{A^*(X^x,Z)}\left[\int_0^\infty \gamma_s^{\theta^*(Z)}e^{-\delta s}ds\right] \\ &\leq \frac{\sigma_{\max}}{\delta}\left\|\frac{\varphi}{\sigma}\right\|_\infty =: L, \end{aligned}$$

where in the last inequality, we use the martingale property of $(\gamma_s^{\theta^*(Z)})_s$. Therefore, we obtain $|v(x,y)| \leq L + U^{-1}(y)$. □

The following result is a verification theorem. It consists in proving that, given a smooth solution to the HJBVI (4.4), this candidate coincides with value function. It allows us to obtain the optimal control in a feedback form.

**Theorem 4.1.** *We suppose that there exist a constant $\hat{b} > 0$ and a continuous function $w : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ such that*:

(i) $w(x,0) = 0, w \in C^2(\mathbb{R}\times[0,\hat{b}))$ *satisfying the growth condition* (4.6).
(ii) $w > -U^{-1}$ *on* $\mathbb{R}\times(0,\hat{b})$ *and* $w = -U^{-1}$ *on* $\mathbb{R}\times[\hat{b},\infty)$.
(iii) $\delta w(x,y) - \sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+}\{\mathcal{L}^{a,r}w(x,y) + \varphi(a) - \kappa\mathbb{1}_{\{a>0\}} - r\} = 0$ *for all* $(x,y) \in \mathbb{R}\times(0,\hat{b})$.
(iv) $\delta(-U^{-1}(y)) - \sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+}\{\mathcal{L}^{a,r}(-U^{-1}(y)) + \varphi(a) - \kappa\mathbb{1}_{\{a>0\}} - r\} \geq 0$ *for all* $(x,y)\in\mathbb{R}\times[\hat{b},\infty)$.

*We also assume that*

$$\sup_{(R,\tau,A^*(Z))\in\mathcal{Y}} \mathbb{E}[|e^{-\delta\tau}U^{-1}(J_\tau^{C,y})|^2] < \infty. \tag{4.7}$$

*Then we have*:

(1) *For all* $(x,y)\in\mathbb{R}\times\mathbb{R}^+$, *we have* $w(x,y) \geq v(x,y)$.
(2) *Suppose that there exist two measurable non-negative functions* $(a^*,r^*)$ *defined on* $\mathbb{R}^+\times\mathbb{R}^+$ *s.t. for all* $y\in(0,\hat{b})$, *we have*

$$\begin{aligned} &\sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+}\{\mathcal{L}^{a,r}w(x,y) - \kappa\mathbb{1}_{\{a>0\}} + \varphi(a) - r\} \\ &\quad = \mathcal{L}^{a^*(x,y),r^*(x,y)}w(x,y) + \varphi(a^*(x,y)) - \kappa\mathbb{1}_{\{a^*(x,y)>0\}} - r^*(x,y) \end{aligned}$$

*and the SDE*

$$\begin{aligned} dJ_t^{C,y} &= [\lambda J_t^{C,y} - U(r^*(X_t^x,J_t^{C,y})) - \psi(a^*(X_t^x,J_t^{C,y}),X_t^x,Z_t) \\ &\qquad + |Z_t|\kappa]dt + Z_t dW_t, \\ J_0^{C,y} &= y, \end{aligned} \tag{4.8}$$

*admits a unique solution* $\widehat{J_t^C}$. *We define*

$$\tau^* := \inf\{t \geq 0 \ : \ w(X, \widehat{J_t^C}) \leq -U^{-1}(\widehat{J_t^C})\}. \tag{4.9}$$

*We assume that* $(r^*(X, \widehat{J^C}), \tau^*, a^*(X, \widehat{J^C}))$ *lies in* $\mathcal{Y}$ *and* $\mathbb{E}^{\mathbb{P}}[e^{(\rho-2\lambda)\tau^*} \times \widehat{J_{\tau^*}^C}^2 \mathbb{1}_{\{\tau^*<\infty\}}] < \infty$.

*If* $w$ *is a solution of* (4.4) *with boundary condition given by* (4.5), *then* $w = v$, *and* $\tau^*$ *is an optimal stopping time for the problem* (4.3).

**Proof.** (1) Let $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ and an admissible control $(R, \tau, A^*(X^x, Z)) \in \mathcal{Y}$. If $y = 0$, then from assumption (i), we have $v(x, 0) = w(x, 0) = 0$. We assume that $0 < y$. From (iii) and (iv), we have

$$\delta w(x, y) \geq \sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+} \{\mathcal{L}^{a,r} w(x, y) + \varphi(a) - \kappa \mathbb{1}_{\{a>0\}} - r\}. \tag{4.10}$$

For $n \in \mathbb{N}$, We introduce the stopping time $\tau_n$:

$$\tau_n := \tau \wedge \inf \left\{ t \geq 0, \sigma(X_t^x) \left| \frac{\partial w(X_t^x, J_t^{C,y})}{\partial x} + \frac{\partial w(X_t^x, J_t^{C,y})}{\partial y} \frac{h'(A^*(X_t^x, Z_t))}{\varphi'(A^*(X_t^x, Z_t))} \mathbb{1}_{\{A^*(X_t^x,Z_t)>0\}} \right| \geq n \right\}.$$

From (i)–(ii), we have $w$ is continuous on $\mathbb{R} \times \mathbb{R}^+$, $w \in C^2(\mathbb{R} \times [0, \hat{b}))$ and $w = -U^{-1} \in C^2([\hat{b}, \infty))$, then $w$ is continuous and piece-wise $C^2$ on $\mathbb{R} \times \mathbb{R}^+$. Applying the generalized Itô's formula (see Krylov [10, Theorem 2, p. 124]) to the process $e^{-\delta t}\gamma_t^{\theta^*(Z)} w(X_t^x, J_t^{C,y})$ between 0 and $\tau_n$, using inequality (4.10) and Bayes formula, we obtain

$$\begin{aligned} w(x, y) &\geq \mathbb{E}^{A^*(X^x,Z)} \left[ \gamma_{\tau_n}^{\theta^*(Z)} \int_0^{\tau_n} e^{-\delta s} (\varphi(A^*(X_s^x, Z_s)) \right. \\ &\qquad \left. - \kappa \mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s) ds + e^{-\delta\tau_n} \gamma_{\tau_n}^{\theta^*(Z)} w(X_{\tau_n}^x, J_{\tau_n}^{C,y}) \right] \\ &= \mathbb{E} \left[ \gamma_{\tau_n}^{A^*,\theta^*} \int_0^{\tau_n} e^{-\delta s} (\varphi(A^*(X_s^x, Z_s)) - \kappa \mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s) ds \right. \\ &\qquad \left. + e^{-\delta\tau_n} \gamma_{\tau_n}^{A^*,\theta^*} w(X_{\tau_n}^x, J_{\tau_n}^{C,y}) \right], \end{aligned} \tag{4.11}$$

where $\gamma_{\tau_n}^{A^*,\theta^*} = \gamma_{\tau_n}^{A^*(X^x,Z)} \gamma_{\tau_n}^{\theta^*(Z)}$. In the next step, we show that the sequence

$$\left( \gamma_{\tau_n}^{A^*,\theta^*} \int_0^{\tau_n} e^{-\delta s} (\varphi(A^*(X_s^x, Z_s)) - \kappa \mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s) ds + e^{-\delta\tau_n} \gamma_{\tau_n}^{A^*,\theta^*} w(X_{\tau_n}^x, J_{\tau_n}^{C,y}) \right)_n$$

is uniformly integrable under $\mathbb{P}$. Let $p \in (1,2)$, we define $p_1 := \frac{2}{2-p}$, $p_2 := \frac{2}{p}$ the conjugate of $p_1$. We denote by $\hat{p} := pp_1 \in (2,\infty)$. Using Hölder's inequality, and the growth condition of $w$, we obtain

$$\begin{aligned}
&\mathbb{E}[|e^{-\delta\tau_n}\gamma_{\tau_n}^{A^*,\theta^*} w(X_{\tau_n}^x, J_{\tau_n}^{C,y})|^p] \\
&\quad \le \mathbb{E}[|Le^{-\delta\tau_n}\gamma_{\tau_n}^{A^*,\theta^*} + e^{-\delta\tau_n}\gamma_{\tau_n}^{A^*,\theta^*} U^{-1}(J_{\tau_n}^{C,y})|^p] \\
&\quad \le C\mathbb{E}[|\gamma_{\tau_n}^{A^*,\theta^*}|^p + |e^{-\delta\tau_n}\gamma_{\tau_n}^{A^*,\theta^*} U^{-1}(J_{\tau_n}^{C,y})|^p] \\
&\quad \le C(\mathbb{E}[|\gamma_{\tau_n}^{A^*,\theta^*}|^p] + \mathbb{E}[|\gamma_{\tau_n}^{A^*,\theta^*}|^{\hat{p}}]^{\frac{1}{p_1}} \mathbb{E}[|e^{-\delta\tau_n} U^{-1}(J_{\tau_n}^{C,y})|^{pp_2}]^{\frac{1}{p_2}}) \\
&\quad \le C(\mathbb{E}[|\gamma_{\tau}^{A^*,\theta^*}|^p] + \mathbb{E}[|\gamma_{\tau}^{A^*,\theta^*}|^{\hat{p}}]^{\frac{1}{p_1}} \mathbb{E}[|e^{-\delta\tau_n} U^{-1}(J_{\tau_n}^{C,y})|^{2}]^{\frac{1}{p_2}}),
\end{aligned}$$

where $C$ is a generic positive constant which could change from line to line.

From Jensen's inequality, we have $\mathbb{E}[|\gamma_{\tau}^{A^*,\theta^*}|^p] \le (\mathbb{E}[|\gamma_{\tau}^{A^*,\theta^*}|^{\hat{p}}])^{\frac{1}{p_1}}$. From inequality (4.6) and assumption (4.7), we deduce

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|e^{-\delta\tau_n}\gamma_{\tau_n}^{A^*,\theta^*} w(X_{\tau_n}^x, J_{\tau_n}^{C,y})|^p] < \infty.$$

Using the same arguments as above, we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\gamma_{\tau_n}^{A^*,\theta^*}\int_0^{\tau_n} e^{-\delta s}(\varphi(A^*(X_s^x,Z_s)) - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds\right|^p\right] \\
&\quad \le \mathbb{E}[|\gamma_{\tau_n}^{A^*,\theta^*}|^{pp_1}]^{\frac{1}{p_1}}\mathbb{E}\left[\left|\int_0^{\tau_n} e^{-\delta s}(\varphi(A^*(X_s^x,Z_s))\right.\right. \\
&\qquad \left.\left. - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds\right|^{pp_2}\right]^{\frac{1}{p_2}} \\
&\quad \le C\mathbb{E}[|\gamma_{\tau}^{A^*,\theta^*}|^{\hat{p}}]^{\frac{1}{p_1}}\mathbb{E}\left[\int_0^{\infty} e^{-\delta s}ds\int_0^{\tau} e^{-\delta s}(\varphi(A^*(X_s^x,Z_s))\right. \\
&\qquad \left. - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)^2 ds\right]^{\frac{1}{p_2}} \\
&\quad \le C\mathbb{E}[|\gamma_{\tau}^{A^*,\theta^*}|^{\hat{p}}]^{\frac{1}{p_1}}\mathbb{E}\left[\int_0^{\tau} e^{-\delta s}(\varphi(A^*(X_s^x,Z_s))^2\right. \\
&\qquad \left. + \kappa^2\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} + R_s^2)ds\right]^{\frac{1}{p_2}},
\end{aligned}$$

where the second inequality is obtained by using Hölder's inequality and since $\int_0^{\tau_n} e^{-\delta s}ds \le \int_0^{\infty} e^{-\delta s}ds$ a.s. As $(R,\tau,A^*(X^x,Z)) \in \mathcal{Y}$, we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\left|\gamma_{\tau_n}^{A^*,\theta^*}\int_0^{\tau_n} e^{-\delta s}(\varphi(A^*(X_s^x,Z_s)) - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds\right|^p\right] < \infty.$$

According to Theorems A.1.2 and A.1.1 in Pham [12], we have the convergence of the previous sequences in $L^1(\mathbb{P})$. By passing to the limit in (4.11), we obtain

$$\begin{aligned} w(x,y) \geq \mathbb{E}^{A^*(X^x,Z)}\Bigg[&\gamma_\tau^{\theta^*(Z)}\int_0^\tau e^{-\delta s}(\varphi(A^*(X_s^x,Z_s)) \\ &- \kappa \mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds - e^{-\delta\tau}\gamma_\tau^{\theta^*(Z)}U^{-1}(J_\tau^{C,y})\Bigg] \end{aligned}$$

and so for all $(x,y) \in \mathbb{R}\times(0,\infty)$, we obtain $w(x,y) \geq v(x,y)$ and then for all $(x,y)\in\mathbb{R}\times\mathbb{R}^+$ we have

$$w(x,y) \geq v(x,y). \tag{4.12}$$

(2) We fix $(x,y)\in\mathbb{R}\times(0,\infty)$. Let $\tau_n$ be the stopping time given by

$$\begin{aligned} \tau_n = \tau^* \wedge \inf\Bigg\{t\geq 0, \sigma(X_t^x)\Bigg|&\frac{\partial w(X_t^x,\hat{J}_t^{C,y})}{\partial x} \\ &+ \frac{\partial w(X_t^x,\hat{J}_t^{C,y})}{\partial y}\frac{h'(a^*(X_t^x,\hat{J}_t^{C,y}))}{\varphi'(a^*(X_t^x,\hat{J}_t^{C,y}))}\mathbb{1}_{\{a^*(X_t^x,\hat{J}_t^{C,y})>0\}}\Bigg| \geq n\Bigg\}. \end{aligned}$$

Since $[\![0,\tau_n[\![\ \subset [\![0,\tau^*[\![$, then by (ii), $w(X_t^x,\hat{J}_t^{C,y}) > -U^{-1}(\hat{J}_t^{C,y})$ on $[\![0,\tau_n[\![$, and so by (iii), we have

$$\begin{aligned} &\delta w(X_t^x,\hat{J}_t^{C,y}) - \mathcal{L}^{a^*(X_t^x,\hat{J}_t^{C,y}),r^*(X_t^x,\hat{J}_t^{C,y})}w(X_t^x,\hat{J}_t^{C,y}) \\ &\quad - (\varphi(a^*(X_t^x,\hat{J}_t^{C,y})) - \kappa\mathbb{1}_{\{a^*(X_t^x,\hat{J}_t^{C,y})>0\}} - r^*(X_t^x,\hat{J}_t^{C,y})) = 0 \text{ on } [\![0,\tau_n[\![. \end{aligned}$$

Therefore,

$$\begin{aligned} w(x,y) = \mathbb{E}^{a^*(X^x,\hat{J}^{C,y})}\Bigg[&\gamma_{\tau_n}^{\theta^*(Z)}\int_0^{\tau_n} e^{-\delta s}(\varphi(a^*(X_s^x,\hat{J}_s^{C,y})) - \kappa\mathbb{1}_{\{a^*(X_s^x,\hat{J}_s^{C,y})>0\}} \\ &- a^*(X_s^x,\hat{J}_s^{C,y}))ds + e^{-\delta\tau_n}\gamma_{\tau_n}^{\theta^*(Z)}w(X_{\tau_n}^x,\hat{J}_{\tau_n}^{C,y})\Bigg]. \end{aligned}$$

Using the same previous steps, we can pass to the limit. From the definition of $\tau^*$ and since $\tau_n \longrightarrow \tau^*$ when $n$ goes to infinity we obtain

$$\begin{aligned} w(x,y) = \mathbb{E}^{a^*(X^x,\hat{J}^{C,y})}\Bigg[&\gamma_{\tau^*}^{\theta^*(Z)}\int_0^{\tau^*} e^{-\delta s}(\varphi(a^*(X_s^x,\hat{J}_s^{C,y})) - \kappa\mathbb{1}_{\{a^*(X_s^x,\hat{J}_s^{C,y})>0\}} \\ &- a^*(X_s^x,\hat{J}_s^{C,y}))ds - e^{-\delta\tau^*}\gamma_{\tau^*}^{\theta^*(Z)}U^{-1}(\hat{J}_{\tau^*}^{C,y})\Bigg] \\ \leq\ & v(x,y). \end{aligned}$$

As $w(x,0) = v(x,0)$ for all $x\in\mathbb{R}$, we deduce that for all $(x,y)\in\mathbb{R}\times\mathbb{R}^+$, we have $w(x,y)\leq v(x,y)$. Combining with inequality (4.12), we deduce that $w = v$. □

**Remark 4.1.** The optimal rent depends on the partial derivative of $u$ with respect to $y$ and is given by $r^*(x,y) = (U')^{-1}(-\frac{1}{\frac{\partial v(x,y)}{\partial y}})\mathbf{1}_{\frac{\partial v(x,y)}{\partial y}>0}$.

## 5. Numerical Study

In this section, we solve the HJBVI (4.4) numerically. First, we localize the domain to $[-\bar{x},\bar{x}]$, $[0,\bar{y}]$, where $\bar{x}$, $\bar{y}$ are two empirical boundaries. Then, we discretize (4.4), by using finite difference approximations. Finally, we solve the discrete variational inequality obtained, by Howard algorithm.

### 5.1. *Numerical scheme*

To discretize the HJBVI (4.4), we adopt the finite difference method. We replace the domain by a bounded domain $[-\bar{x},\bar{x}]$, $[0,\bar{y}]$. Let $\Delta_x$ (respectively, $\Delta_y$) be the finite difference step on the state coordinate and $x^{\Delta} = (x_i)_{i=1,N}, x_i = -\bar{x} + i\Delta_x$ (respectively, $y^{\Delta} = (y_i)_{i=1,N}, y_i = i\Delta_y$), are the points of the grid $\Omega_{\Delta} := \{-\bar{x} + i\Delta_x, i = 1,\dots,N\} \times \{j\Delta_y, j = 1,\dots,N\}$.

We approximate the derivatives of $v$ as follows:

$$\frac{\partial v(x,y)}{\partial x} \simeq \begin{cases} \dfrac{v(x+\Delta_x,y) - v(x,y)}{\Delta_x} & \text{if } -\varphi(a) + \kappa \mathbb{1}_{\{a>0\}} \geq 0, \\ \dfrac{v(x,y) - v(x-\Delta_x,y)}{\Delta_x} & \text{if not,} \end{cases}$$

$$\frac{\partial v(x,y)}{\partial y} \simeq \begin{cases} \dfrac{v(x,y+\Delta_y) - v(x,y)}{\Delta_y} & \text{if } -\lambda y + U(r) - h(a) \geq 0, \\ \dfrac{v(x,y) - v(x,y-\Delta_y)}{\Delta_y} & \text{if not,} \end{cases}$$

$$\frac{\partial v^2(x,y)}{\partial x^2} \simeq \frac{v(x+\Delta_x,y) - 2v(x,y) + v(x-\Delta_x,y)}{\Delta_x^2},$$

$$\frac{\partial v^2(x,y)}{\partial y^2} \simeq \frac{v(x,y+\Delta_y) - 2v(x,y) + v(x,y-\Delta_y)}{\Delta_y^2},$$

$$\begin{aligned}\frac{\partial v^2(x,y)}{\partial x\partial y} \simeq{}& \frac{v(x+\Delta_x,y+\Delta_y) - v(x+\Delta_x,y) - v(x,y+\Delta_y) + 2v(x,y)}{2\Delta_x\Delta_y} \\ &- \frac{v(x-\Delta_x,y) + v(x,y-\Delta_y) - v(x-\Delta_x,y-\Delta_y)}{2\Delta_x\Delta_y},\end{aligned}$$

$$v(x,0) = 0, \quad v(x,\bar{y}) = -U^{-1}(\bar{y}), \quad \frac{\partial v(-\bar{x},y)}{\partial x} = 0, \quad \frac{\partial v(\bar{x},y)}{\partial x} = 0.$$

We denote $\mathcal{L}^{\Delta,a,r}$ the linear operator, which is a block tridiagonal matrix of dimension

$$(N-1)^2 \times (N-1)^2 [\mathcal{L}^{\Delta,(a,r)}]_{k,k-1} = \bar{A}_k^{\Delta,(a,r)};$$
$$[\mathcal{L}^{\Delta,(a,r)}]_{k,k} = D_k^{\Delta,(a,r)};$$
$$[\mathcal{L}^{\Delta,(a,r)}]_{k,k+1} = A_k^{\Delta,(a,r)}, \quad k = 1, \dots, N-1;$$

$\bar{A}_k^{\Delta,(a,r)}, D_k^{\Delta,(a,r)}, A_k^{\Delta,(a,r)}$ are tridiagonal matrices $(N-1)\times(N-1)$ defined by

$$[\bar{A}_k^{\Delta,(a,r)}]_{i,i-1} = \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}; \quad [\bar{A}_k^{\Delta,(a,r)}]_{i,i} = \frac{\beta_{k,i}^{1,-}}{\Delta_x} + \frac{\alpha_{k,i}^1}{\Delta_x^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y};$$

$$[\bar{A}_k^{\Delta,(a,r)}]_{i,i+1} = 0; \quad [D_k^{\Delta,(a,r)}]_{i,i-1} = \frac{\beta_{k,i}^{2,-}}{\Delta_y} + \frac{\alpha_{k,i}^2}{\Delta_y^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y};$$

$$[D_k^{\Delta,(a,r)}]_{i,i} = \delta - \frac{|\beta_{k,i}^1|}{\Delta_x} - \frac{|\beta_{k,i}^2|}{\Delta_y} - 2\left(\frac{\alpha_{k,i}^1}{\Delta_x^2} + \frac{\alpha_{k,i}^2}{\Delta_y^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}\right);$$

$$[D_k^{\Delta,(a,r)}]_{i,i+1} = \frac{\beta_{k,i}^{2,+}}{\Delta_y} + \frac{\alpha_{k,i}^2}{\Delta_y^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}; \quad [A_k^{\Delta,(a,r)}]_{i,i-1} = 0;$$

$$[A_k^{\Delta,(a,r)}]_{i,i} = \frac{\beta_{k,i}^{1,+}}{\Delta_x} + \frac{\alpha_{k,i}^1}{\Delta_x^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y};$$

$$[A_k^{\Delta,(a,r)}]_{i,i+1} = \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}, \quad i = 1, \dots, N-1;$$

where $\beta_{i,j}^{l,+}(x,y) = \max(\beta_{i,j}^l, 0)$, $\beta_{i,j}^{l,-} = \max(-\beta_{i,j}^l, 0)$, $l = 1, 2$, and the parameters $\gamma$, $\beta^1$, $\beta^2$, $\alpha^1$, $\alpha^2$ are given as

$$\beta_{i,j}^1 = -\varphi(a) + \kappa\mathbf{1}_{\{a>0\}}, \quad \beta_{i,j}^2 = -\lambda y - h(a) + U(r),$$

$$\alpha_{i,j}^1 = -\frac{1}{2}\sigma^2(x_i)\mathbf{1}_{a>0}, \quad \alpha_{i,j}^2 = -\frac{1}{2}\left(\sigma(x_i)\frac{h'(a)}{\varphi'(a)}\right)^2\mathbf{1}_{a>0},$$

$$\gamma_{i,j} = -\sigma^2(x_i)\frac{h'(a)}{\varphi'(a)}\mathbf{1}_{a>0}.$$

We obtain $(N-1)^2$ linear approximation equations with $(N-1)^2$ unknowns $V^\Delta = (v_1^\Delta, \dots, v_{N-1}^\Delta)$ where $v_i^\Delta = (v_{i,1}^\Delta, \dots, v_{i,N-1}^\Delta)$,

$$\min\left[\inf_{(r,a)\in\mathbb{R}_+\times\mathbb{R}_+}[\mathcal{L}^{\Delta,(a,r)}V^\Delta + B^{\Delta,(a,r)}], V^\Delta + U^{-1}(y^\Delta)\right] = 0, \tag{5.1}$$

where $U^{-1}(y^{\Delta}) := (U^{-1}(y_i^{\Delta}))_{i=1,\ldots,N-1}$ and for $i \in \{1,\ldots,N-1\}$, the $\mathbb{R}^{(N-1)}$ vector $U^{-1}(y_i^{\Delta})$ is defined by $U^{-1}(y_i^{\Delta}) := (U^{-1}(y_i),\ldots,U^{-1}(y_i))$ and

$$\begin{aligned} B^{\Delta,(a,r)} = (&B_1^{\Delta,(a,r)} + \bar{A}_1^{\Delta,(a,r)} v_1^{\Delta}, B_2^{\Delta,(a,r)},\ldots,\\ &B_{N-2}^{\Delta,(a,r)}, B_{N-1}^{\Delta,(a,r)} + A_{N-1}^{\Delta,(a,r)} v_{N-1}^{\Delta}), \end{aligned}$$

such that

$$B_k^{\Delta,\kappa,(a,r)} = \begin{pmatrix} -\varphi(a)+\kappa+r+[\bar{A}_k^{\Delta,(a,r)}]_{1,0} v_{k-1,0}^{\Delta} + [D_k^{\Delta,(a,r)}]_{1,0} v_{k,0}^{\Delta} \\ -\varphi(a)+\kappa+r \\ \vdots \\ -\varphi(a)+\kappa+r \\ -\varphi(a)+\kappa+r+[D_k^{\Delta,(a,r)}]_{N-1,N} v_{k,N}^{\Delta} + [A_k^{\Delta,(a,r)}]_{N-1,N} v_{k+1,N}^{\Delta} \end{pmatrix}.$$

To solve the discrete variational inequality (5.1), we used Howard's algorithm. We describe it as follows.

**The Howard algorithm:** It consists in computing iteratively two sequences $((a_{i,j}^n, r_{i,j}^n)_{i,j=1,\ldots N-1})_{n\geq 1}$ and $(V^{\Delta,n})_{n\geq 1}$, as follows:

- Step $2n-1$. To the vector $V^{\Delta,n}$, we associate the strategy

$$(a^n, r^n) \in \arg\min_{a,r}\{\mathcal{L}^{\Delta,(a,r)} V^{\Delta,n} + B^{\Delta,(a,r)}\}.$$

- Step $2n$. From the strategy $(a^n, r^n)$, we compute a partition $(D_1^n \cup D_2^n)$ of $\mathbb{R}_+^2$ defined by

$$\begin{aligned} \mathcal{L}^{\Delta,(a^n,r^n)} V^{\Delta,n} + B^{\Delta,(a^n,r^n)} &\leq V^{\Delta,n} + U^{-1}(y^{\Delta}), \text{ on } D_1^n,\\ \mathcal{L}^{\Delta,(a^n,r^n)} V^{\Delta,n} + B^{\Delta,(a^n,r^n)} &> V^{\Delta,n} + U^{-1}(y^{\Delta}), \text{ on } D_2^n. \end{aligned}$$

  The solutions $V^{\Delta,n+1}$ is obtained by solving two linear systems

$$\begin{aligned} \mathcal{L}^{\Delta,(a^n,r^n)} V^{\Delta,n+1} + B^{\Delta,(a^n,r^n)} &= 0, \text{ on } D_1^n,\\ V^{\Delta,n+1} + U^{-1}(y^{\Delta}) &= 0, \text{ on } D_2^n. \end{aligned}$$

If $|V^{\Delta,n+1} - V^{\Delta,n}| \leq \varepsilon$, stop, otherwise, go to step $2n+1$.

### 5.2. *Numerical results*

For the numerical implementation, we choose the following functions for $\varphi$ (the impact of the effort on the social welfare), $h$ (the cost of effort) and the consortium's

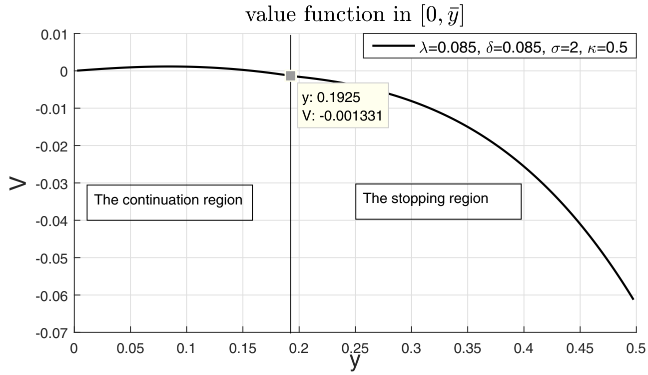


Fig. 1. Value function in $[0, \bar{y}]$.

utility $U$:

$$\varphi(x) = 3(1 - \exp(-\alpha x)), \quad h(x) = \exp(\beta x) - 1 \quad \text{and}$$

$$U(x) = x^p; \quad \alpha = 0.1, \ \beta = 0.1, \ p = \frac{1}{3} \quad \text{and} \quad \sigma = 2.$$

The preference parameters for the public and the consortium are, respectively, $\delta = \lambda = 0.085$. We study numerically the impact of the ambiguity by varying $\kappa$: $\kappa = 0, \ 0.25, \ 0.5$ or 1. We start from $v(0) = 0$ and we take $\overline{y} = 0.5$.

Figure 1 represents the value function on $[0, \bar{y}]$, for $\kappa = 0.5$.

Figures 2–4 represent, respectively, the value function, the optimal effort and the optimal rent for different values of $\kappa$.

Figure 5 represents the optimal rent as function of the optimal effort.

We observe in Fig. 1 that the value function is concave, in accordance with Sannikov [13], Dumav and Riedel [4] and Hajjej *et al.* [7].

Figure 2 plots the value function for different values of $\kappa$. We observe that the ambiguity influences the continuation region. The higher of $\kappa$, the smaller the continuation region. In particular, the value function of the public without ambiguity is higher than the one with ambiguity. Figures 3 and 4, respectively, plot the optimal effort and the optimal rent on the continuation region for different values of $\kappa$. When $\kappa$ is higher, the consortium makes more effort to compensate the ambiguity and to increase the social welfare. However, at a high level of the consortium's value function, the public pays a higher rent when the ambiguity decreases.

Figure 5 shows the relationship between the optimal effort and the optimal rent.

The optimal rent versus the optimal effort is a decreasing function, which is in accordance with the results of Sannikov [13] and Dumav and Riedel [4]. It is more costly for the public to incentivize the consortium to start to work. In fact for a low level of the effort, the Principal pays more the Agent to make an effort. Figure 5 shows that, in the presence of ambiguity, the public stops to pay rent when the level of the effort exceeds 7.3. On the other hand, from Fig. 6, we remark that in

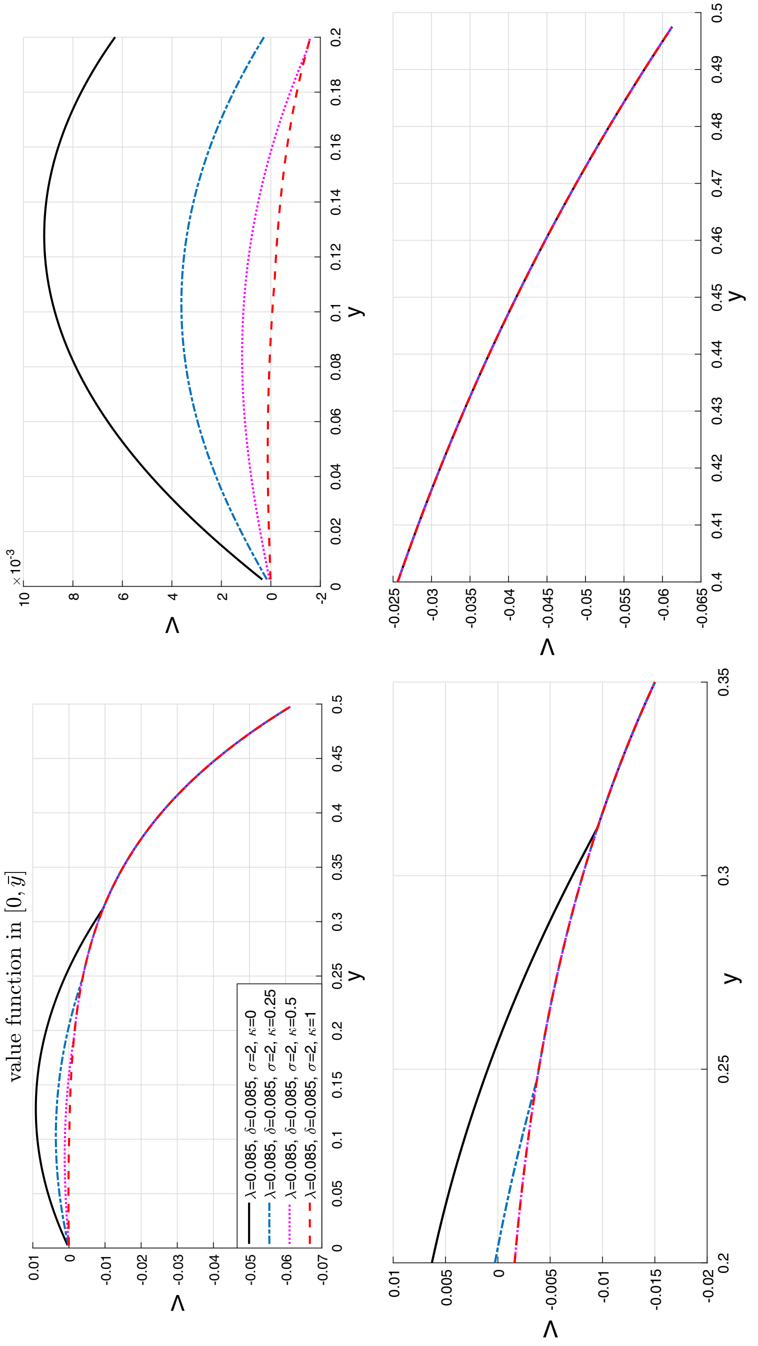


Fig. 2. Value function for $\sigma = 2$ and different $\kappa$.

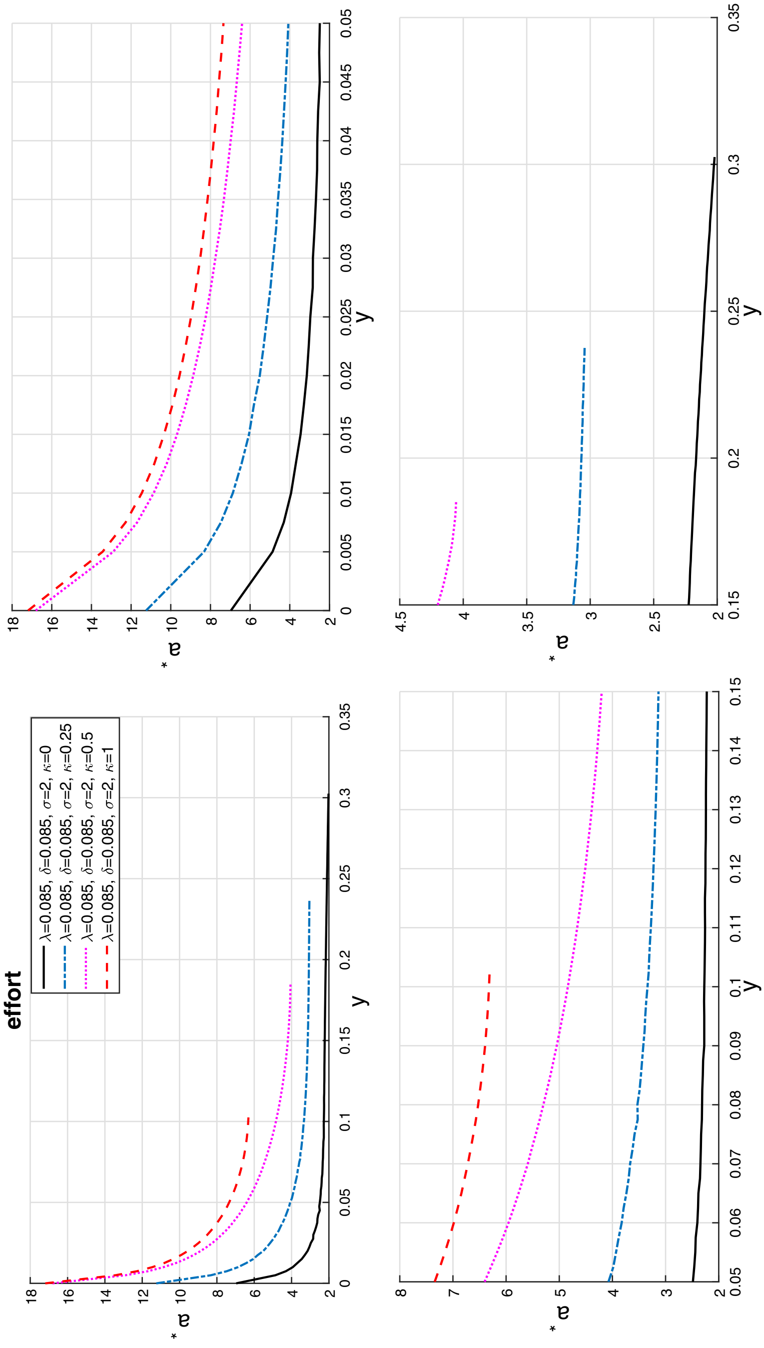


Fig. 3. Optimal effort for $\sigma = 2$ and different $\kappa$.

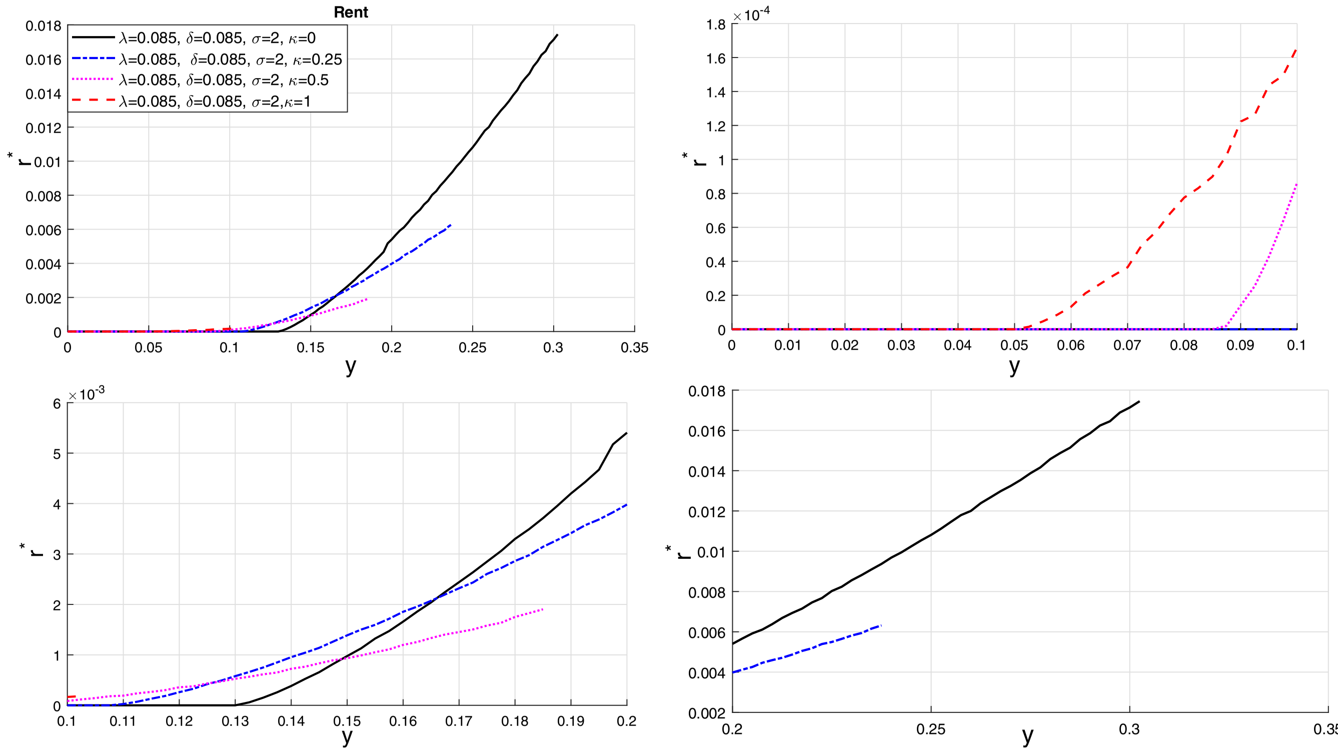


Fig. 4. Optimal rent for $\sigma = 2$ and different $\kappa$.

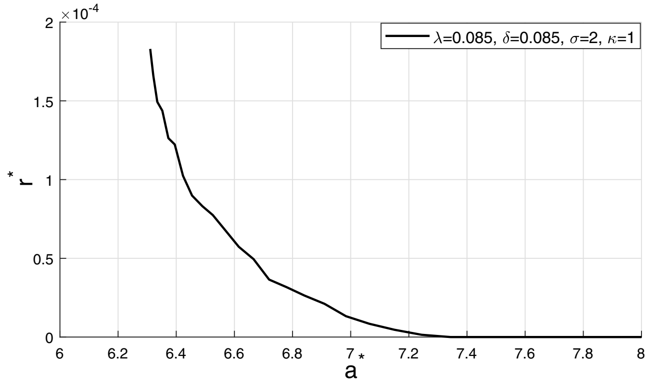


Fig. 5. Optimal rent versus optimal effort with ambiguity.

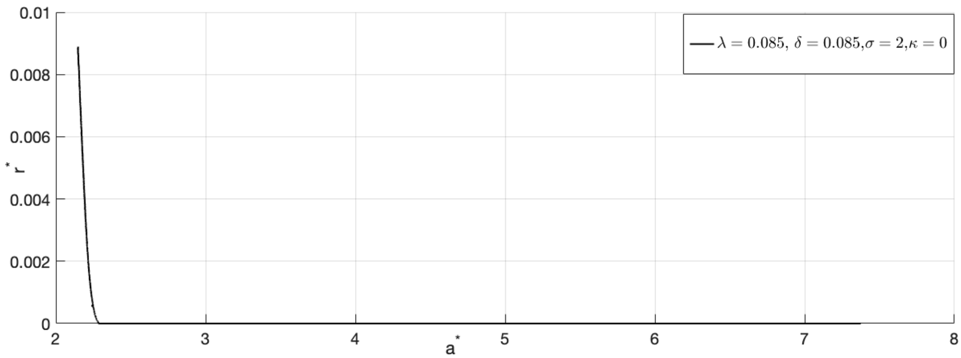


Fig. 6. Optimal rent versus optimal effort without ambiguity.

the absence of ambiguity, the public stops to pay a rent from the level 2.25. From Figs. 5 and 6, when the consortium receives a rent, we notice that its level is higher when there is no ambiguity.

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