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NONLINEAR FRACTIONAL Q-DIFFERENTIAL EQUATIONS INVOLVING HILFER-KATUGAMPOLA DERIVATIVES OF MOVING ORDERS

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Abstract. This study comprehensively investigates the existence, uniqueness, and stability of solutions for nonlinear fractional q-differential equations involving Hilfer-Katugampola q-derivatives of moving orders. We apply the Banach contraction principle and Schauder's fixed-point theorem to establish the existence of solutions. Furthermore, we examine the stability of the solutions using Ulam-Hyers theorems. Two detailed examples are provided to illustrate the practical applicability and validity of our theoretical results.

Keywords. q-calculus; q-Hilfer-Katugampola fractional derivative; existence; uniqueness; Ulam-Hyers stability.

AMS (MOS) subject classification: 05A30; 26A33; 33D05; 34K37; 39A13.

1 Introduction

Fractional calculus, which extends the classical concepts of differentiation and integration to non-integer orders, has garnered significant attention due to its powerful ability to model memory and hereditary properties in complex systems. It has found widespread applications in various scientific and engineering domains, including blood flow dynamics, electrical circuits, biology, chemistry, physics, control theory, wave propagation, and signal and image processing. For comprehensive insights into its practical applications, readers are referred to the works of Afshari et al. [1, 2, 3], Agrawal [4], Basti *et al.* [5, 6, 7, 8], Benchohra *et al.* [9, 10, 11, 12, 13], Herrmann [14], Hilfer [15], and Kilbas *et al.* [16].

In parallel, the twentieth century witnessed a revolutionary development in quantum mechanics, which inspired the emergence of quantum calculus, a framework introduced by Jackson in 1909 ([17]). This branch of calculus, which avoids the traditional concept of limits, is deeply influential in mathematics, mechanics, and physics [18, 19, 20]. Recognizing its potential, researchers such as Al-Salam and Agarwal extended the theory to fractional q-calculus, a synthesis of fractional and quantum calculus, to better model physical, biological, and economic systems [21, 22].

As a result, a significant body of research has emerged focusing on fractional q-differential equations (q-FDEs), which offer refined modeling capabilities in systems governed by nonlocality and discrete structures. For instance, Salim et al. [23] analyzed the fractional q-difference problem

$$\begin{cases} ({}^{c}D_q^{\zeta}\xi)(\varsigma) = \wp(\varsigma,\xi(\varsigma));\ \varsigma \in \Psi := [0,\beta], \\ \xi(0) = \xi_0 \in F, \end{cases}$$

where ${}^{c}D_q^{\zeta}$ denotes the Caputo fractional q-difference operator of order $\zeta \in (0,1]$ and F is a Banach space. In a related contribution, the existence of solutions to an implicit q-fractional problem in Banach algebras was proven in [24]:

$$\begin{cases} {}^{c}D_q^{\zeta}\left(\frac{\xi(\varsigma)}{h(\varsigma,\xi(\varsigma))}\right) = \psi\left(\varsigma,\xi(\varsigma),{}^{c}D_q^{\zeta}\left(\frac{\xi(\varsigma)}{h(\varsigma,\xi(\varsigma))}\right)\right);\ \varsigma \in \Psi := [0,\beta], \\ \xi(0) = \xi_0 \in \mathbb{R}. \end{cases}$$

More recently, the importance of nonlinear fractional q-differential equations has grown due to their ability to characterize complex systems with greater fidelity.

In [25], the existence and uniqueness of solutions for the following Cauchy-type q-fractional problem of the form

$$\begin{cases} \mathcal{D}_{q,a^+}^{\delta,\eta} x\left(t\right) = \psi\left(t, x\left(t\right)\right), & n-1<\delta\leq n,\ n\in\mathbb{N},\ 0\leq\eta\leq 1, \\ \lim\limits_{t\to a^+}\left(D_q^k\ \mathcal{J}_q^{(n-\delta)(1-\eta)}x\right)(t) = c_k\in\mathbb{R}, & k=0,\ 1,\ldots,\ n-1, \end{cases}$$

were studied. Here $\mathcal{D}_{q,a^+}^{\delta,\eta}$ is the Hilfer fractional q-derivative of order δ, and $\psi : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, for $0<a<b<\infty$.

This work aims to study the existence, uniqueness, and stability of solutions for a problem of nonlinear fractional q-differential equations involving Hilfer-Katugampola q-derivatives of moving orders in the Banach space $\mathcal{L}_{q,\rho}^1\left([a,b],\mathbb{C}\right)$ with initial conditions.

The problem under consideration is written as follows:

$$^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}\left[x\left(t\right)-h\left(t\right)\right] = \psi\left(t, x\left(t\right), {}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}\left[x\left(t\right)-h\left(t\right)\right]\right), \tag{1}$$

where $t\in\Omega=[a,b]$, for some reals $b>a>0$, with the following conditions

$$^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1}x\left(a\right)=\omega\in\mathbb{C}, \quad \text{and} \quad {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}x\left(a\right)=0,\ \text{for } k=2,\ 3,\ldots,\ n. \tag{2}$$

Here ${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}$ and ${}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}$ are the q-analogue of Hilfer-Katugampola fractional derivatives of order α and β respectively, with $0\leq\eta\leq 1$, $m-1<\beta\leq m\in\mathbb{N}$, $\theta=\beta+\eta\left(m-\beta\right)$, $\max\left\{\theta,n-1\right\}<\alpha\leq n$ for $n\in\mathbb{N}-\{1\}$, and $\gamma=\alpha+\eta\left(n-\alpha\right)$. Also, $0<q<1$, $\rho>0$, and $h\in\mathcal{L}_{q,\rho}^1\left(\Omega,\mathbb{C}\right)$ is a complex-valued function given by

$$h\left(t\right)=\frac{\omega\left(t^{\rho}-a^{\rho}\right)_{q^{\rho}}^{(\gamma-1)}}{\left[\rho\right]_q^{\gamma-1}\Gamma_{q^{\rho}}\left(\gamma\right)}. \tag{3}$$

We mention that the operator ${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha}$ presents the Katugampola fractional q-derivative of order α, while $\psi : \Omega \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear complex-valued function.

We impose the following hypotheses:

$(\mathcal{H}_1)$ $\psi : \Omega \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear complex-valued function such that

$$\psi\left(\cdot, x(\cdot), {}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}\left[x(\cdot) - h(\cdot)\right]\right) \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C}), \quad x, h \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C}),$$

and there exist two nonnegative constants c_1 and c_2 such that the function ψ satisfies

$$|\psi(t, x, y) - \psi(t, \tilde{x}, \tilde{y})| \leq c_1 |x - \tilde{x}| + c_2 |y - \tilde{y}|,$$

for any $x, y, \tilde{x}, \tilde{y} \in \mathbb{C}$.

$(\mathcal{H}_2)$ There exist three nonnegative functions $u_1 \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{R}_+)$ and $(u_i)_{i=2,3} \in \mathcal{C}(\Omega, \mathbb{R}_+)$, such that the function ψ satisfies

$$|\psi(t, x, y)| \leq u_1(t) + u_2(t)|x| + u_3(t)|y|,$$

for any $x, y \in \mathbb{C}$ and each $t \in \Omega$.

We denote

$$u_1^* = \|u_1\|_{\mathcal{L}_{q,\rho}^{1}}, \quad u_2^* = \sup_{t\in\Omega} |u_2(t)|, \quad u_3^* = \sup_{t\in\Omega} |u_3(t)|, \quad \text{and } \lambda_\alpha = \frac{\left(b^\rho - (qa)^\rho\right)_{q^\rho}^{(\alpha)}}{[\rho]_q^\alpha \, \Gamma_{q^\rho}(\alpha + 1)}.$$

For simplifying the writing, we substitute λ with λ_α, and we use κ instead of $\lambda_{\alpha-\beta}$.

2 Necessary Definitions and Preliminaries

This section provides definitions and lemmas for some q-calculus concepts that will be used in this paper (see [2, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]).

Let $0 < q < 1$ and $\alpha \in \mathbb{C}$, we define the q-integer $[\alpha]_q$ by

$$[\alpha]_q = \begin{cases} \frac{1-q^\alpha}{1-q}, & q \neq 1, \\ \alpha, & q = 1, \end{cases}$$

then

$$[\rho\alpha]_q = \frac{1 - q^{\rho\alpha}}{1 - q} = \frac{1 - q^{\rho\alpha}}{1 - q^\rho}\frac{1 - q^\rho}{1 - q} = [\rho]_q [\alpha]_{q^\rho}.$$

The definition of q-factorial $[n]_q!$ is

$$[n]_q! = \begin{cases} [n]_q \times [n-1]_q \times \cdots \times [1]_q, & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases}$$

We introduce the q-shifted as follows

$$(t-a)_q^{(0)}=1,\qquad (t-a)_q^{(n)}=\prod_{k=0}^{n-1}\left(t-aq^k\right),\text{ for } n\in\mathbb{N}\text{ and } 0\leq a\leq t.$$

The q-shifted is also introduced for $\alpha\in\mathbb{C}$ with $\alpha\notin\mathbb{N}$ by

$$(t-a)_q^{(\alpha)}=t^\alpha\prod_{k=0}^{\infty}\frac{t-q^ka}{t-q^{\alpha+k}a},\quad 0\leq a\leq t.$$

For $\alpha\in\mathbb{C}\setminus\{-n,\ n\in\mathbb{N}\cup\{0\}\}$, the q-Gamma function is defined by

$$\Gamma_q(\alpha)=\frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},\quad 0<q<1.$$

Obviously

$$\Gamma_q(1)=1,\ \Gamma_q(n+1)=[n]_q!\quad\text{and}\quad\Gamma_q(\alpha+1)=[\alpha]_q\,\Gamma_q(\alpha).$$

The q-derivative of a function x is defined by

$$D_q^0x(t)=x(t),\quad\text{and}\quad D_qx(t)=\frac{x(t)-x(qt)}{(1-q)\,t},\quad t\neq 0,$$

and $D_qx(0)=\lim_{t\to 0}D_qx(t)$. Also, the q-derivative of higher order is given by

$$\left(D_q^mx\right)(t)=D_qD_q^{m-1}x(t),\quad m\in\mathbb{N}.$$

The following formulations with respect to t and a have q-derivatives that can be expressed as

$$_tD_q\left[(t^\rho-a^\rho)_{q^\rho}^{(\alpha)}\right]=t^{\rho-1}[\rho\alpha]_q\,(t^\rho-a^\rho)_{q^\rho}^{(\alpha-1)},\tag{4}$$

and

$$_aD_q\left[(t^\rho-a^\rho)_{q^\rho}^{(\alpha)}\right]=-a^{\rho-1}[\rho\alpha]_q\,(t^\rho-(qa)^\rho)_{q^\rho}^{(\alpha-1)}.\tag{5}$$

Additionally, reversing the integration's order is provided by

$$\int_a^b\int_a^t x(\tau)\,d_q\tau d_qt=\int_a^b\int_{q\tau}^b x(\tau)\,d_qtd_q\tau.\tag{6}$$

When $q\to 1$, the foregoing results are equivalent to those in ordinary.

For $\rho\in\mathbb{R}$, $0<q<1$, we define the Banach space

$$\mathcal{L}_{q,\rho}^1(\Omega,\mathbb{C})=\left\{x:\Omega\to\mathbb{C},\ \|x\|_{\mathcal{L}_{q,\rho}^1}<\infty\right\},$$

with

$$\|x\|_{\mathcal{L}_{q,\rho}^1}=\int_a^b t^{\rho-1}\,|x(t)|\,d_qt.\tag{7}$$

Now, we give some definitions of q-fractional operators introduced in [28, 29, 26, 27], with a little change in the notation.

Definition 1. *Let $\alpha > 0$, $0 < q < 1$, and $\rho > 0$, be such that $x \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$, the left-sided fractional q-integral of Katugampola is defined by*

$$^{\rho}\mathcal{J}^{\alpha}_{q,a^+}x(t) = \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \int_a^t \tau^{\rho-1} \left(t^\rho - (q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)} x(\tau)\, d_q\tau.$$

Lemma 2. *The Katugampola fractional q-integral $^{\rho}\mathcal{J}^{\alpha}_{q,a^+}$ is well defined and bounded in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$, with*

$$\left\| ^{\rho}\mathcal{J}^{\alpha}_{q,a^+}x \right\|_{\mathcal{L}^1_{q,\rho}} \leq \lambda \|x\|_{\mathcal{L}^1_{q,\rho}},$$

for any $\alpha > 0$, $0 < q < 1$, and $\rho > 0$.

Proof. Let $x \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$. Employing (4) and (6) gives us

$$\begin{aligned}
\left\| ^{\rho}\mathcal{J}^{\alpha}_{q,a^+}x \right\|_{\mathcal{L}^1_{q,\rho}} &= \int_a^b t^{\rho-1} \left| \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \int_a^t \tau^{\rho-1} \left(t^\rho - (q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)} x(\tau)\, d_q\tau \right| d_q t \\
&\leq \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)} \int_a^b \tau^{\rho-1} |x(\tau)| \left[\int_{q\tau}^b t^{\rho-1} \left(t^\rho - (q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)} d_q t \right] d_q\tau \\
&\leq \frac{[\rho]_q^{1-\alpha}}{[\rho\alpha]_q \Gamma_{q^\rho}(\alpha)} \int_a^b \tau^{\rho-1} |x(\tau)| \left[\int_{q\tau}^b {}_tD_q \left(t^\rho - (q\tau)^\rho\right)_{q^\rho}^{(\alpha)} d_q t \right] d_q\tau \\
&\leq \frac{1}{[\rho]_q^{\alpha} \Gamma_{q^\rho}(\alpha+1)} \int_a^b \tau^{\rho-1} |x(\tau)| \left(b^\rho - (q\tau)^\rho\right)_{q^\rho}^{(\alpha)} d_q\tau \\
&\leq \frac{\left(b^\rho - (qa)^\rho\right)_{q^\rho}^{(\alpha)}}{[\rho]_q^{\alpha} \Gamma_{q^\rho}(\alpha+1)} \int_a^b \tau^{\rho-1} |x(\tau)|\, d_q\tau \\
&\leq \lambda \|x\|_{\mathcal{L}^1_{q,\rho}}.
\end{aligned}$$

The proof is complete. □

Definition 3. *Let $n - 1 < \alpha \leq n \in \mathbb{R}$, $0 < q < 1$, and $\rho > 0$, then the left-sided fractional q-derivative of Katugampola is defined by*

$$\begin{aligned}
^{\rho}\mathcal{D}^{\alpha}_{q,a^+}x(t) &= \left(t^{1-\rho}D_q\right)^n {}^{\rho}\mathcal{J}^{n-\alpha}_{q}x(t) \\
&= \frac{[\rho]_q^{1-n+\alpha}}{\Gamma_{q^\rho}(n-\alpha)} \left(t^{1-\rho}D_q\right)^n \int_a^t \tau^{\rho-1} \left(t^\rho - (q\tau)^\rho\right)_{q^\rho}^{(n-\alpha-1)} x(\tau)\, d_q\tau, \qquad (8)
\end{aligned}$$

provided that $x \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$, with $\left(^{\rho}\mathcal{D}^{0}_{q,a^+}x\right)(t) = x(t)$.

The following results are given in [26] with consideration that $0 < q < 1$ and $\rho > 0$.

Lemma 4. *Let $\alpha, \beta > 0$, then the semi-group property for the Katugampola fractional q-integral is given by*

$$\left({}^{\rho}\mathcal{J}_q^{\alpha}\ {}^{\rho}\mathcal{J}_q^{\beta}x\right)(t) = \left({}^{\rho}\mathcal{J}_q^{\alpha+\beta}x\right)(t), \quad \forall x \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C}).$$

Lemma 5. *For $n-1 < \alpha \leq n \in \mathbb{N}$, and $x \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C})$, we get*

$$\left({}^{\rho}\mathcal{D}_{q,a^+}^{\alpha}\ {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha}x\right)(t) = x(t), \quad \forall t \in \Omega.$$

Lemma 6. *Let $n-1 < \alpha \leq n \in \mathbb{N}$, be such that $x \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C})$, then*

$$\left({}^{\rho}\mathcal{J}_{q,a^+}^{\alpha}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha}x\right)(t) = x(t) - \sum_{k=1}^{n} \frac{[\rho]_q^{k-\alpha}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha-k}x(a)}{\Gamma_{q^\rho}(\alpha-k+1)} (t^\rho - a^\rho)_{q^\rho}^{(\alpha-k)}.$$

Lemma 7. *Assume $\beta \geq \alpha \geq 0$, if $x \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C})$, then*

$$\left({}^{\rho}\mathcal{D}_{q,a^+}^{\alpha}\ {}^{\rho}\mathcal{J}_{q,a^+}^{\beta}x\right)(t) = {}^{\rho}\mathcal{J}_{q,a^+}^{\beta-\alpha}x(t), \quad \forall t \in \Omega.$$

Moreover, if ${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha-\beta}x(t)$ exists and $\alpha \geq \beta \geq 0$, then for all $t \in \Omega$, we obtain

$$\left({}^{\rho}\mathcal{D}_{q,a^+}^{\alpha}\ {}^{\rho}\mathcal{J}_{q,a^+}^{\beta}x\right)(t) = {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha-\beta}x(t).$$

Definition 8 (Hilfer-Katugampola fractional q-derivative [27])**.** *Let $n-1 < \alpha \leq n \in \mathbb{N}$ and $0 \leq \eta \leq 1$. The q-analogue of left-sided Hilfer-Katugampola fractional derivative ${}^{\rho}\mathcal{D}_{q,a^{\pm}}^{\alpha,\eta}$ is defined by*

$${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}x(t) = \left({}^{\rho}\mathcal{J}_{q,a^+}^{\eta(n-\alpha)} \left(t^{1-\rho}D_q\right)^n\ {}^{\rho}\mathcal{J}_{q,a^+}^{(1-\eta)(n-\alpha)}x\right)(t).$$

Here ${}^{\rho}\mathcal{J}_{q,a^+}^{\alpha}$ presents the Katugampola fractional q-integral given by Definition 1.

The derivative ${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}$ can be expressed in terms of the Katugampola fractional q-integral ${}^{\rho}\mathcal{J}_{q,a^+}^{\alpha}$ and q-derivative ${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha}$ as follows

$$\begin{aligned} {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta} &= {}^{\rho}\mathcal{J}_{q,a^+}^{\eta(n-\alpha)} \left(t^{1-\rho}D_q\right)^n\ {}^{\rho}\mathcal{J}_{q,a^+}^{n-\gamma} \\ &= {}^{\rho}\mathcal{J}_{q,a^{\pm}}^{\eta(n-\alpha)}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma} \end{aligned}$$

where $\gamma = \alpha + \eta(n-\alpha)$. Consequently, ${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}$ is well defined in $\mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C})$.

Remark 9. *The q-analogue of Hilfer-Katugampola fractional derivative ${}^{\rho}\mathcal{D}_{q,a^{\pm}}^{\alpha,\eta}$ becomes*

1. *The q-analogue of Hilfer fractional derivative when $\rho \to 1$.*
2. *The q-analogue of Hilfer-Hadamard fractional derivative when $\rho \to 0^+$.*

3. *The q-analogue of Katugampola fractional derivative (8) for $\eta = 0$, which generalizes*

   (i) *Riemann-Liouville type fractional q-derivative when $\rho \to 1$.*

   (ii) *Hadamard type fractional q-derivative when $\rho \to 0^+$.*

4. *The q-analogue of Caputo-Katugampola fractional derivative [29] for $\eta = 1$, which generalizes*

   (i) *Caputo type fractional q-derivative when $\rho \to 1$.*

   (ii) *Caputo-Hadamard type fractional q-derivative when $\rho \to 0^+$.*

The following results are in [26, 27].

Lemma 10. *Let $\delta \in \mathbb{C}$ be such that $Re(\delta) > -1$, then*

$$\left({}^{\rho}\mathcal{J}_{q,a^+}^{\alpha} \left(t^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\delta)}\right)(t) = \frac{\Gamma_{q^{\rho}}(\delta + 1)}{[\rho]_q^{\alpha}\,\Gamma_{q^{\rho}}(\alpha + \delta + 1)} \left(t^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\alpha+\delta)},$$

and

$$\left({}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta} \left(t^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\delta)}\right)(t) = \left({}^{\rho}\mathcal{D}_{q,a^+}^{\alpha} \left(t^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\delta)}\right)(t) = \frac{[\rho]_q^{\alpha}\,\Gamma_{q^{\rho}}(\delta + 1)}{\Gamma_{q^{\rho}}(\delta - \alpha + 1)} \left(t^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\delta-\alpha)}, \tag{9}$$

hold for every $\alpha, \rho > 0$ and $0 < q < 1$.

Lemma 11. *Let $n - 1 < \alpha \leq n \in \mathbb{N}$, $0 \leq \eta \leq 1$, $0 < q < 1$, and $\rho > 0$, be such that $\gamma = \alpha + \eta\,(n - \alpha)$. If $x \in \mathcal{L}_{q,\rho}^{1}(\Omega, \mathbb{C})$, then*

$$\begin{aligned}\left({}^{\rho}\mathcal{J}_{q,a^+}^{\alpha}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta} x\right)(t) &= \left({}^{\rho}\mathcal{J}_{q,a^+}^{\gamma}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma} x\right)(t) \\ &= x(t) - \sum_{k=1}^{n} \frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}_q^{\gamma-k} x(a)}{\Gamma_{q^{\rho}}(\gamma - k + 1)} \left(t^{\rho} - a^{\rho}\right)_{q^{\rho}}^{(\gamma-k)}.\end{aligned}$$

Also, if ${}^{\rho}\mathcal{D}_{q,a^+}^{\eta(n-\alpha)} x$ exists, then

$${}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}\ {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha} x = {}^{\rho}\mathcal{J}_{q,a^+}^{\eta(n-\alpha)}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\eta(n-\alpha)} x. \tag{10}$$

3 Existence and Uniqueness of Solutions

Throughout the rest of this paper, we put $0 \leq \eta \leq 1$, $m - 1 < \beta \leq m \in \mathbb{N}$, $\theta = \beta + \eta\,(m - \beta)$, and

$$\max\{\theta, n - 1\} < \alpha \leq n \in \mathbb{N} - \{1\}.$$

Also $\gamma = \alpha + \eta\,(n - \alpha)$, $0 < q < 1$, and $\rho > 0$.

In this following, we present some lemmas to illustrate our main results.

Lemma 12. *We have*

1. *The complex-valued function h given by (3) satisfies ${}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1}h(t)=\omega$, and can be expressed by*

$$h(t)=\sum_{k=1}^{n}\frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}_q^{\gamma-k}h(a)}{\Gamma_{q^\rho}(\gamma-k+1)}(t^\rho-a^\rho)_{q^\rho}^{(\gamma-k)}.$$

2. *We have*

$$\left({}^{\rho}\mathcal{J}_{q,a^+}^{\gamma}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma}h\right)(t)=0,\quad \forall t\in\Omega. \tag{11}$$

Proof. We apply (9) for $\alpha=\gamma-k$ and $\delta=\gamma-1$, we get

$$\begin{aligned}{}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}h(t) &= \frac{\omega}{[\rho]_q^{\gamma-1}\Gamma_{q^\rho}(\gamma)}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}(t^\rho-a^\rho)_{q^\rho}^{(\gamma-1)}\\ &= \frac{\omega}{[\rho]_q^{\gamma-1}\Gamma_{q^\rho}(\gamma)}\left(\frac{[\rho]_q^{\gamma-k}\Gamma_{q^\rho}(\gamma-1+1)}{\Gamma_{q^\rho}(\gamma-1-(\gamma-k)+1)}(t^\rho-a^\rho)_{q^\rho}^{(\gamma-1-(\gamma-k))}\right)\\ &= \frac{\omega}{[\rho]_q^{k-1}\Gamma_{q^\rho}(k)}(t^\rho-a^\rho)_{q^\rho}^{(k-1)}.\end{aligned}$$

1. For $k=1$, we obtain ${}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1}h(t)=\omega$, and ${}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}h(a)=0$ for each $k=2,3,\ldots,n$. Then

$$h(t)=\sum_{k=1}^{n}\frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}_q^{\gamma-k}h(a)}{\Gamma_{q^\rho}(\gamma-k+1)}(t^\rho-a^\rho)_{q^\rho}^{(\gamma-k)}.$$

2. Similarly, Lemma 6 implies

$$\begin{aligned}\left({}^{\rho}\mathcal{J}_{q,a^+}^{\gamma}\ {}^{\rho}\mathcal{D}_{q,a^+}^{\gamma}h\right)(t) &= h(t)-\sum_{k=1}^{n}\frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}_q^{\gamma-k}h(a)}{\Gamma_{q^\rho}(\gamma-k+1)}(t^\rho-a^\rho)_{q^\rho}^{(\gamma-k)}\\ &= h(t)-h(t)\\ &= 0.\end{aligned}$$

The proof is complete. □

Lemma 13. *Let $x\in\mathcal{L}_{q,\rho}^{1}(\Omega,\mathbb{C})$, then problem (1)–(2) is equivalent to the q-integral equation*

$$x(t)=h(t)+{}^{\rho}\mathcal{J}_{q,a^+}^{\alpha}\varphi(t),\quad t\in\Omega, \tag{12}$$

where $\varphi\in\mathcal{L}_{q,\rho}^{1}(\Omega,\mathbb{C})$ satisfies

$$\varphi(t)=\psi\left(t,h(t)+{}^{\rho}\mathcal{J}_{q,a^+}^{\alpha}\varphi(t),\ {}^{\rho}\mathcal{J}_{q,a^+}^{\alpha-\beta}\varphi(t)\right).$$

Proof. Let $x \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ satisfies problem (1)–(2). Starting applying ${}^{\rho}\mathcal{J}^{\alpha}_{q,a}$ on both sides (1), we obtain

$$ {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\left[x(t) - h(t)\right] = {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi\left(t, x(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}\left[x(t) - h(t)\right]\right). \tag{13} $$

As

$$ \begin{aligned} {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\left[x(t) - h(t)\right] &= {}^{\rho}\mathcal{J}^{\gamma}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{q,a^+}\left[x(t) - h(t)\right] \\ &= {}^{\rho}\mathcal{J}^{\gamma}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{q,a^+}x(t) - {}^{\rho}\mathcal{J}^{\gamma}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\gamma}_{q,a^+}h(t). \end{aligned} $$

After using (11) from Lemma 12, and employing Lemma 11, we get

$$ \mathcal{J}^{\alpha}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\left[x(t) - h(t)\right] = x(t) - \sum_{k=1}^{n} \frac{[\rho]_q^{k-\gamma}\ {}^{\rho}\mathcal{D}^{\gamma-k}_{q}x(a)}{\Gamma_{q^\rho}(\gamma - k + 1)}\left(t^\rho - a^\rho\right)^{(\gamma-k)}_{q^\rho}. $$

Substituting in (13) and using conditions (2), we obtain

$$ x(t) = h(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t), $$

where $\varphi(t) = \psi\left(t, x(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}\left[x(t) - h(t)\right]\right)$. In addition,

$$ {}^{\rho}\mathcal{D}^{\gamma-1}_{q,a^+}x(t) = \omega + {}^{\rho}\mathcal{D}^{\eta(n-\alpha)-1}_{q,a^+}\varphi(t), \quad \text{for every } t \in \Omega. $$

As ${}^{\rho}\mathcal{D}^{\gamma-1}_{q,a^+}x(a) = \omega$, we deduce that

$$ {}^{\rho}\mathcal{D}^{\eta(n-\alpha)-1}_{q,a^+}\varphi(a) = 0. \tag{14} $$

Since $\beta + \eta(m - \beta) = \theta < \alpha$, employing Lemma 7 allows us to write

$$ \begin{aligned} {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}\left[x(t) - h(t)\right] &= {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}\left[h(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t) - h(t)\right] \\ &= {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t) \\ &= {}^{\rho}\mathcal{J}^{\eta(m-\beta)}_{q,a^+}\ {}^{\rho}\mathcal{D}^{\theta}_{q,a^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t) \\ &= {}^{\rho}\mathcal{J}^{\eta(m-\beta)}_{q,a^+}\ {}^{\rho}\mathcal{J}^{\alpha-\beta-\eta(m-\beta)}_{q,a^+}\varphi(t) \\ &= {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi(t), \end{aligned} $$

then we can define $\varphi \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ as a nonlinear complex-valued function that is satisfying the functional equation

$$ \varphi(t) = \psi(t, \omega + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t), {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi(t)). $$

Otherwise, assume that $x \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ satisfies equation (12), next, we prove that x satisfies problem (1)–(2). We apply the operator ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}$ Hilfer-Katugampola q-derivative on both sides of equation (12), we obtain

$$ {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\left[x(t) - h(t)\right] = {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}\ {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi(t), $$

from (10), Lemma 6, and (14), we have

$$\begin{aligned} {}^{\rho}\mathcal{D}_{q,a^{+}}^{\alpha,\eta}\ {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi(t) &= {}^{\rho}\mathcal{J}_{q,a^{+}}^{\eta(n-\alpha)}\ {}^{\rho}\mathcal{D}_{q,a^{+}}^{\eta(n-\alpha)}\varphi(t) \\ &= \varphi(t) - \frac{[\rho]_q^{1-\eta(n-\alpha)}\ {}^{\rho}\mathcal{D}_{q,a^{+}}^{\eta(n-\alpha)-1}\varphi(a)}{\Gamma_{q^{\rho}}(\eta(n-\alpha))}(t^{\rho}-a^{\rho})_{q^{\rho}}^{(\eta(n-\alpha)-1)} \\ &= \varphi(t), \end{aligned}$$

then we get equation (1).

Now, we show that conditions (2) hold. Applying ${}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}$ in both sides of (12), we get

$$\begin{aligned} {}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}x(t) &= {}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}\left[h(t) + {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha}\varphi(t)\right] \\ &= {}^{\rho}\mathcal{D}_{q,a^{+}}^{\gamma-k}h(t) + {}^{\rho}\mathcal{J}_{q,a^{+}}^{k-\eta(n-\alpha)}\varphi(t), \quad \forall k \in \overline{1,n}. \end{aligned}$$

For $t=a$, we arrive to conditions (2). The proof is complete. □

Theorem 14. *Assume the hypotheses* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold. If we put* $\kappa \in \left(0, \min\left(\frac{1}{c_2}, \frac{1}{u_3^*}\right)\right)$ *and*

$$\frac{\lambda u_2^*}{1-\kappa u_3^*} < 1. \tag{15}$$

Then problem (1)–(2) has a least one solution in $\mathcal{L}_{q,\rho}^{1}(\Omega,\mathbb{C})$.

Proof. First, we will transform problem (1)–(2) into a fixed point problem, we define the operator

$$\mathcal{B}x(t) = h(t) + \frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^{\rho}}(\alpha)}\int_a^t \tau^{\rho-1}(t^{\rho}-(q\tau)^{\rho})_{q^{\rho}}^{(\alpha-1)}\varphi(\tau)\,d_q\tau, \tag{16}$$

where $\varphi = \psi\left(t, x, {}^{\rho}\mathcal{J}_{q,a^{+}}^{\alpha-\beta}\varphi\right) \in \mathcal{L}_{q,\rho}^{1}(\Omega,\mathbb{C})$. Therefore $\mathcal{B}x$ is an element of $\mathcal{L}_{q,\rho}^{1}(\Omega,\mathbb{C})$ equipped with the norm

$$\|\mathcal{B}x\|_{\mathcal{L}_{q,\rho}^{1}} = \int_a^b t^{\rho-1}|\mathcal{B}x(t)|\,d_qt.$$

As $(\mathcal{H}_1)$ holds, we notice that if $x \in \mathcal{L}_{q,\rho}^{1}(\Omega,\mathbb{C})$, then $\mathcal{B}$ is a continuous operator as demonstrated in step1.

Next, we demonstrate that $\mathcal{B}$ satisfies the assumptions of Schauder's fixed point theorem, this could be proved through three steps.

a. **$\mathcal{B}$ is a continuous operator**

Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim\limits_{n\to\infty} x_n = x$ in $\mathcal{L}_{q,\rho}^{1}(\Omega,\mathbb{C})$.

After using Lemma 2, we get

$$\begin{aligned}\|\mathcal{B}x_n-\mathcal{B}x\|_{\mathcal{L}^1_{q,\rho}} &= \left\|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi_n-{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi\right\|_{\mathcal{L}^1_{q,\rho}}\\ &= \left\|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\left(\varphi_n-\varphi\right)\right\|_{\mathcal{L}^1_{q,\rho}}\\ &\leq \lambda\left\|\varphi_n-\varphi\right\|_{\mathcal{L}^1_{q,\rho}},\end{aligned}$$

where

$$\begin{cases}\varphi_n(t)=\psi\left(t,x_n(t),{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi_n(t)\right),\\ \varphi(t)=\psi\left(t,x(t),{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi(t)\right).\end{cases}$$

Hypothesis $(\mathcal{H}_1)$ implies

$$\begin{aligned}\|\varphi_n-\varphi\|_{\mathcal{L}^1_{q,\rho}} &= \left\|\psi(t,\ x_n,{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi_n)-\psi(t,\ x,{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi)\right\|_{\mathcal{L}^1_{q,\rho}}\\ &\leq c_1\|x_n-x\|_{\mathcal{L}^1_{q,\rho}}+c_2\left\|{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\left(\varphi_n-\varphi\right)\right\|_{\mathcal{L}^1_{q,\rho}},\end{aligned}$$

then

$$\|\varphi_n-\varphi\|_{\mathcal{L}^1_{q,\rho}}\leq\frac{c_1}{1-\kappa c_2}\|x_n-x\|_{\mathcal{L}^1_{q,\rho}}.$$

Then

$$\|\mathcal{B}x_n-\mathcal{B}x\|_{\mathcal{L}^1_{q,\rho}}\leq\frac{\lambda c_1}{1-\kappa c_2}\|x_n-x\|_{\mathcal{L}^1_{q,\rho}}.$$

Since $x_n\to x$, when $n\to\infty$, then $\|\mathcal{B}x_n-\mathcal{B}x\|_{\mathcal{L}^1_{q,\rho}}\to 0$ as $n\to\infty$. Hence

$$\lim_{n\to\infty}\|\mathcal{B}x_n-\mathcal{B}x\|_{\mathcal{L}^1_{q,\rho}}=0,$$

this implies the continuity of $\mathcal{B}$.

b. $\mathcal{B}$ is defined from a bounded, closed, and convex subset into itself

Using (15), we define

$$r\geq\left(\frac{|\omega|\left(b^{\rho}-a^{\rho}\right)^{(\gamma)}_{q^{\rho}}}{[\rho]^{\gamma}_q\,\Gamma_{q^{\rho}}(\gamma+1)}+\frac{\lambda u_1^*}{1-\kappa u_3^*}\right)\left(\frac{1-\kappa u_3^*}{1-\kappa u_3^*-\lambda u_2^*}\right)$$

and define the subset P_r as follows

$$P_r=\left\{x\in\mathcal{L}^1_{q,\rho}(\Omega,\ \mathbb{C}),\quad \|x\|_{\mathcal{L}^1_{q,\rho}}\leq r\right\}.$$

It is clear that P_r is bounded closed and convex subset of $\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$. Let $\mathcal{B}:P_r\to\mathcal{L}^1_{q,\rho}(\Omega,\mathbb{C})$ be the integral operator defined by (16), then

$\mathcal{B}(P_r) \subset P_r$.
Indeed, by employing $(\mathcal{H}_2)$, we get

$$\begin{aligned} \|\varphi\|_{\mathcal{L}^1_{q,\rho}} &= \left\|\psi(t,x,{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi)\right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \|u_1\|_{\mathcal{L}^1_{q,\rho}} + \sup_{t\in\Omega}|u_2(t)|\,\|x\|_{\mathcal{L}^1_{q,\rho}} + \sup_{t\in\Omega}|u_3(t)|\left\|{}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi\right\|_{\mathcal{L}^1_{q,\rho}}. \end{aligned}$$

Therefore

$$\|\varphi\|_{\mathcal{L}^1_{q,\rho}} \leq u_1^* + u_2^*\|x\|_{\mathcal{L}^1_{q,\rho}} + \kappa u_3^*\|\varphi\|_{\mathcal{L}^1_{q,\rho}},$$

next

$$\|\varphi\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{u_1^* + u_2^* r}{1-\kappa u_3^*}. \tag{17}$$

Consequently

$$\begin{aligned} \|\mathcal{B}x\|_{\mathcal{L}^1_{q,\rho}} &= \left\|h + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi\right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \|h\|_{\mathcal{L}^1_{q,\rho}} + \left\|{}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\varphi\right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \frac{|\omega|\,(b^\rho-a^\rho)^{(\gamma)}_{q^\rho}}{[\rho]_q^\gamma\,\Gamma_{q^\rho}(\gamma+1)} + \lambda\|\varphi\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \frac{|\omega|\,(b^\rho-a^\rho)^{(\gamma)}_{q^\rho}}{[\rho]_q^\gamma\,\Gamma_{q^\rho}(\gamma+1)} + \lambda\left(\frac{u_1^*+u_2^*r}{1-\kappa u_3^*}\right) \\ &\leq \frac{\left(\frac{|\omega|(b^\rho-a^\rho)^{(\gamma)}_{q^\rho}}{[\rho]_q^\gamma\Gamma_{q^\rho}(\gamma+1)} + \frac{\lambda u_1^*}{1-\kappa u_3^*}\right)\left(\frac{1-\kappa u_3^*}{1-\kappa u_3^*-\lambda u_2^*}\right)}{\left(\frac{1-\kappa u_3^*}{1-\kappa u_3^*-\lambda u_2^*}\right)} + \frac{\lambda u_2^* r}{1-\kappa u_3^*} \\ &\leq r \end{aligned}$$

then $\mathcal{B}(P_r) \subset P_r$.

c. **$\mathcal{B}(P_r)$ is an equicontinuous subset**
Let $t_1, t_2 \in \Omega$, be such that $t_1 \leq t_2$, and $x \in P_r$. By using (7) and (17), we can obtain

$$\max_{t\in\Omega}|\varphi(t)| \leq \varphi^* = \frac{[\rho]_q\,(u_1^*+u_2^*r)}{(b^\rho-a^\rho)(1-\kappa u_3^*)}.$$

In another hand

$$
\begin{aligned}
& \left|\mathcal{B}x\left(t_2\right)-\mathcal{B}x\left(t_1\right)\right| \qquad (18)\\
\leq\; & \left|h\left(t_2\right)-h\left(t_1\right)\right|+\frac{[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)}\left|\int_a^{t_2}\tau^{\rho-1}\left(t_2^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}\varphi(\tau)\,d_q\tau\right.\\
& \left.-\int_a^{t_1}\tau^{\rho-1}\left(t_1^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}\varphi(\tau)\,d_q\tau\right|\\
\leq\; & \frac{|\omega|\left(\left(t_2^\rho-a^\rho\right)_{q^\rho}^{(\gamma-1)}-\left(t_1^\rho-a^\rho\right)_{q^\rho}^{(\gamma-1)}\right)}{[\rho]_q^{\gamma-1}\Gamma_{q^\rho}(\gamma)}+\frac{\varphi^*[\rho]_q^{1-\alpha}}{\Gamma_{q^\rho}(\alpha)}\times\\
& \left[\int_a^{t_1}\tau^{\rho-1}\left(\left(t_2^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}-\left(t_1^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}\right)d_q\tau+\right.\\
& \left.\int_{t_1}^{t_2}\tau^{\rho-1}\left(t_2^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}d_q\tau\right]. \qquad (19)
\end{aligned}
$$

By applying (5), we get

$$
\begin{aligned}
& \tau^{\rho-1}\left(\left(t_2^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}-\left(t_1^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}\right)\\
& =\frac{-1}{[\rho\alpha]_q}\,{}_\tau D_q\left(\left(t_2^\rho-\tau^\rho\right)_{q^\rho}^{(\alpha)}-\left(t_1^\rho-\tau^\rho\right)_{q^\rho}^{(\alpha)}\right),
\end{aligned}
$$

then

$$
\begin{aligned}
& \int_a^{t_1}\frac{\left(t_2^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}-\left(t_1^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}}{\tau^{1-\rho}}d_q\tau\\
& \leq\frac{\left(t_2^\rho-a^\rho\right)_{q^\rho}^{(\alpha)}-\left(t_1^\rho-a^\rho\right)_{q^\rho}^{(\alpha)}-\left(t_2^\rho-t_1^\rho\right)_{q^\rho}^{(\alpha)}}{[\rho\alpha]_q}.
\end{aligned}
$$

Also

$$
\tau^{\rho-1}\left(t_2^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}d_q\tau=\frac{-1}{[\rho\alpha]_q}\,{}_\tau D_q\left(t_2^\rho-\tau^\rho\right)_{q^\rho}^{(\alpha)}
$$

then

$$
\int_{t_1}^{t_2}\tau^{\rho-1}\left(t_2^\rho-(q\tau)^\rho\right)_{q^\rho}^{(\alpha-1)}d_q\tau\leq\frac{\left(t_2^\rho-t_1^\rho\right)_{q^\rho}^{(\alpha)}}{[\rho\alpha]_q}.
$$

Thus, (18) gives us

$$
\begin{aligned}
\left|\mathcal{B}x\left(t_2\right)-\mathcal{B}x\left(t_1\right)\right| \leq\; & \frac{|\omega|\left(\left(t_2^\rho-a^\rho\right)_{q^\rho}^{(\gamma-1)}-\left(t_1^\rho-a^\rho\right)_{q^\rho}^{(\gamma-1)}\right)}{[\rho]_q^{\gamma-1}\Gamma_{q^\rho}(\gamma)}\\
& +\frac{\varphi^*\left(\left(t_2^\rho-a^\rho\right)_{q^\rho}^{(\alpha)}-\left(t_1^\rho-a^\rho\right)_{q^\rho}^{(\alpha)}\right)}{[\rho]_q^{\alpha}\Gamma_{q^\rho}(\alpha+1)}.
\end{aligned}
$$

The right-hand side of the latter inequality approaches zero as t_1 tends to t_2. Hence

$$\lim_{t_1 \to t_2} \|\mathcal{B}x_{t_2} - \mathcal{B}x_{t_1}\|_{\mathcal{L}^1_{q,\rho}} = 0.$$

As a consequence of steps ***a.***, ***b.***, and ***c.***, and with the aid of the Ascoli-Arzelà theorem, we deduce the continuity of $\mathcal{B}$, its compactness, and its satisfaction of the assumption required by Schauder's fixed point theorem [31]. Consequently, $\mathcal{B}$ possesses a fixed point in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ that solves problem (1)–(2). □

Theorem 15. *Assume the hypothesis $(\mathcal{H}_1)$ holds. If we put $\kappa \in \left(0, \frac{1}{c_2}\right)$ and*

$$\frac{\lambda c_1}{1 - \kappa c_2} < 1, \tag{20}$$

then problem (1)–(2) admits a unique solution in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$.

Proof. Let $x_1, x_2 \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$, then

$$\|\mathcal{B}x_1 - \mathcal{B}x_2\|_{\mathcal{L}^1_{q,\rho}} \leq \lambda \|\varphi_1 - \varphi_2\|_{\mathcal{L}^1_{q,\rho}}$$

where

$$\varphi_i = \psi\left(t, x_i, {}^{\rho}\mathcal{J}^{\alpha-\beta}_{q,a^+}\varphi_i\right) \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C}), \quad \forall i = 1, 2.$$

We have

$$\|\varphi_1 - \varphi_2\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{c_1}{1 - \kappa c_2} \|x_1 - x_2\|_{\mathcal{L}^1_{q,\rho}}.$$

Therefore

$$\|\mathcal{B}x_1 - \mathcal{B}x_2\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{\lambda c_1}{1 - \kappa c_2} \|x_1 - x_2\|_{\mathcal{L}^1_{q,\rho}}.$$

Thus, according to (19), $\mathcal{B}$ is considered a contraction operator.

Banach's contraction principal (see [31]) helps us infer that $\mathcal{B}$ has only one fixed point which is the unique solution of problem (1)–(2). □

4 Ulam-Hyers Stability Results of Solutions

In this section, we use Definitions 16 and 17 to study the stability of equation (1) in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$.

Definition 16. *Equation (1) is Ulam-Hyers stable if there exists a real number $\mu > 0$ such that for each $\varepsilon > 0$ and each solution $y \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ of the inequality*

$$\left| {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[y(t) - h(t)] - \psi\left(t, y(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[y(t) - h(t)]\right)\right| \leq \varepsilon, \tag{21}$$

there exists a solution $x \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ of (1), with

$$\|y - x\|_{\mathcal{L}^1_{q,\rho}} \leq \mu\varepsilon.$$

Definition 17. *Equation (1) is generalized Ulam-Hyers stable if there exists $f \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, with $f(0) = 0$, such that for each solution y, which is in $\mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ of the inequality (20), there exists a solution x of (1) with*

$$\|y - x\|_{\mathcal{L}^1_{q,\rho}} \leq f(\varepsilon).$$

Remark 18. *Ulam-Hyers stability concept ensures that if a function approximately satisfies a given equation, such as inequality (20), then there exists an exact solution close to it. This guarantees that small perturbations do not lead to significant departures from the true solution, thus enhancing the reliability of approximate solutions in practical applications.*

Before proceeding, we present the following remark introduced in [32], followed by a lemma aimed to simplify subsequent calculations.

Remark 19. *If $y \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ is a solution of the inequality (20), then there exists $g \in \mathcal{C}(\Omega, \mathbb{C})$, such that*

1. ${}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[y(t) - h(t)] = \psi\left(t, y(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[y(t) - h(t)]\right) + g(t)$, *for any* $t \in \Omega$,
2. $|g(t)| \leq \varepsilon$, *for all* $t \in \Omega$.

Lemma 20. *If $y \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ is the solution of the inequality (20), then there exists $\varepsilon > 0$ such that y will be the solution of the inequality:*

$$\left| y(t) - v(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi\left(t, y(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[y(t) - h(t)]\right) \right| \leq \ell\varepsilon,$$

with

$$\ell = \frac{(b^\rho - a^\rho)^{(\alpha)}_{q^\rho}}{[\rho]^{\alpha}_q \Gamma_{q^\rho}(\alpha + 1)} \quad \text{and} \quad v(t) = \sum_{k=1}^{n} \frac{[\rho]^{k-\gamma}_q \; {}^{\rho}\mathcal{D}^{\gamma-k}_q y(a)}{\Gamma_{q^\rho}(\gamma - k + 1)} (t^\rho - a^\rho)^{(\gamma-k)}_{q^\rho}.$$

Proof. If $y \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ is a solution of (20). Then from Remark 19, we have

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha,\eta}_{q,a^+}[y(t) - h(t)] = \psi_y(t) + g(t), & t \in \Omega, \\ |g(t)| \leq \varepsilon, & \varepsilon > 0, \end{cases}$$

where $\psi_y(t)$ is a simple notation for $\psi\left(t, y(t), {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}[y(t) - h(t)]\right)$. Hence

$$y(t) = v(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}[\psi_y(t) + g(t)].$$

Then, for all $t \in \Omega$, we get

$$\begin{aligned} & \left| y(t) - v(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi_y(t) \right| \\ = & \left| {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}[\psi_y(t) + g(t)] - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi_y(t) \right| = \left| {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+} g(t) \right| \\ \leq & \frac{[\rho]^{1-\alpha}_q}{\Gamma_{q^\rho}(\alpha)} \int_a^t \tau^{\rho-1} (t^\rho - (q\tau)^\rho)^{(\alpha-1)}_{q^\rho} |g(\tau)| \, d_q\tau. \\ \leq & \ell\varepsilon. \end{aligned}$$

The proof is complete. □

Theorem 21. *Let $x, y \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$, where y is a solution of the inequality (20) and x is the unique solution of equation (1) with the conditions*

$$ {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+}x(a) = {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+}y(a), \text{ for each } k = 1, 2, \ldots, n, $$

such that

$$ \left\| {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}(y-x) \right\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{c_0 - c_1}{c_2} \|y-x\|_{\mathcal{L}^1_{q,\rho}}, \text{ for some } c_0 > c_1. $$

If the assumption $(\mathcal{H}_1)$ holds and $c_0\lambda < 1$, then equation (1) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$, we define $y \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ as a solution of the inequality (20) and $x \in \mathcal{L}^1_{q,\rho}(\Omega, \mathbb{C})$ as the unique solution of equation (1) with the conditions

$$ {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+}x(a) = {}^{\rho}\mathcal{D}^{\gamma-k}_{q,a^+}y(a), \text{ for } k = 1, 2, \ldots, n. $$

Thus

$$ x(t) = v(t) + {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi_x(t). $$

And

$$ \begin{aligned} |y(t) - x(t)| &\leq \left| y(t) - v(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi_y(t) \right| \\ &\quad + \left| {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi_y(t) - {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}\psi_x(t) \right|. \end{aligned} $$

Using Lemma 20 and $(\mathcal{H}_1)$ makes us obtain

$$ \begin{aligned} \|y-x\|_{\mathcal{L}^1_{q,\rho}} &\leq \frac{(b^\rho - a^\rho)\ell}{[\rho]_q}\varepsilon + \left\| {}^{\rho}\mathcal{J}^{\alpha}_{q,a^+}(\psi_y - \psi_x) \right\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \frac{(b^\rho - a^\rho)\ell}{[\rho]_q}\varepsilon + \lambda \|\psi_y - \psi_x\|_{\mathcal{L}^1_{q,\rho}} \\ &\leq \frac{(b^\rho - a^\rho)\ell}{[\rho]_q}\varepsilon + \lambda \left(c_1 \|y-x\|_{\mathcal{L}^1_{q,\rho}} + c_2 \left\| {}^{\rho}\mathcal{D}^{\beta,\eta}_{q,a^+}(y-x) \right\|_{\mathcal{L}^1_{q,\rho}} \right) \\ &\leq \frac{(b^\rho - a^\rho)\ell}{[\rho]_q}\varepsilon + c_0\lambda \|y-x\|_{\mathcal{L}^1_{q,\rho}}. \end{aligned} $$

Then

$$ \|y-x\|_{\mathcal{L}^1_{q,\rho}} \leq \mu\varepsilon. $$

where $\mu = \frac{(b^\rho - a^\rho)\ell}{[\rho]_q(1-c_0\lambda)}$. Definition 16 helps us infer that equation (1) is Ulam-Hyers stable on Ω. This completes the proof.

If we select $f(\varepsilon) = \mu\varepsilon$, it follows that $f(0) = 0$. Subsequently, according to Definition 17, it can be inferred that equation (1) manifests generalized Ulam-Hyers' stability. □

Remark 22. *Ulam-Hyers stability demonstrated above guarantees that small perturbations in the functional or initial conditions do not lead to significant deviations from the exact solution. Moreover, the linear dependence of the stability bound on the perturbation implies generalized Ulam-Hyers stability, which accommodates more flexible, function-dependent perturbations, particularly useful in modeling and data-driven scenarios.*

5 Illustrative Examples

We will now present two examples to demonstrate the key findings of our study.

Example 23. *Consider the following problem*

$$\begin{cases} {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}\left(x-h\right)(t) = \frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha)}|\arctan(t)|\left(2+|x(t)|+\left|{}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}(x-h)(t)\right|\right)}{20q\ \Gamma_{q^\rho}(\alpha+1)\left(1+|x(t)|+\left|{}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}(x-h)(t)\right|\right)}, t\in\left[\frac{1}{2},\frac{3}{2}\right], \\ \left({}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1}x\right)(a) = \omega\in\mathbb{C},\ \text{and}\ \left({}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}x\right)(a) = 0,\ \text{for}\ k=\ 2,\ 3,\ldots,\ n. \end{cases} \tag{22}$$

We set

$$\psi(t,x,y) = \frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha)}|\arctan(t)|(2+|x|+|y|)}{20q\ \Gamma_{q^\rho}(\alpha+1)(1+|x|+|y|)},$$

where $\alpha=\frac{5}{3}$, $\eta=\frac{3}{4}$, $q=\frac{1}{4}$, $\beta=\frac{1}{3}$, *and* $\rho=1$. *Then* $\gamma=\frac{23}{12}$, $\theta=\frac{5}{6}$, *and* $n=2$.

The function ψ is continuous. For any $x,y,\tilde{x},\tilde{y}\in\mathbb{C}$, and $\forall t\in\left[\frac{1}{2},\frac{3}{2}\right]$, we get

$$|\psi(t,x,y)-\psi(t,\tilde{x},\tilde{y})| \leq \frac{(b^\rho - a^\rho)_{q^\rho}^{(\alpha)}\pi}{40q\ \Gamma_{q^\rho}(\alpha+1)}(|x-\tilde{x}|+|y-\tilde{y}|).$$

Hence, the hypothesis $(\mathcal{H}_1)$ is satisfied with $c_1=c_2=\frac{(b^\rho - a^\rho)_{q^\rho}^{(\alpha)}\pi}{40q\ \Gamma_{q^\rho}(\alpha+1)}\simeq 0,33368179$.

We also have

$$|\psi(t,x,y)| \leq \frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha)}|\arctan(t)|(2+|x|+|y|)}{20q\ \Gamma_{q^\rho}(\alpha+1)}.$$

Thus, the hypothesis $(\mathcal{H}_2)$ is satisfied with

$$u_1(t) = \frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha)}|\arctan(t)|}{10q\ \Gamma_{q^\rho}(\alpha+1)}, \quad u_2(t)=u_3(t) = \frac{(t^\rho - a^\rho)_{q^\rho}^{(\alpha)}|\arctan(t)|}{20q\ \Gamma_{q^\rho}(\alpha+1)},$$

and

$$u_1^* = \|u_1\|_{\mathcal{L}^1_{q,\rho}} \leq \frac{(b^\rho - a^\rho)_{q^\rho}^{(\alpha+1)}\pi}{20q\ [\rho]_q\Gamma_{q^\rho}(\alpha+2)} = \frac{\pi}{5\Gamma_{\frac{1}{4}}\left(\frac{11}{3}\right)},$$

also

$$\begin{aligned} u_2^* &= u_3^* = \frac{\left(b^\rho - a^\rho\right)_{q^\rho}^{(\alpha)} \pi}{40q\ \Gamma_{q^\rho}\left(\alpha+1\right)} \simeq 0,33368179 \\ \lambda &= \frac{\left([1]_q\right)^{-\frac{5}{3}} \left(\frac{3}{2}-\frac{1}{2}\right)_q^{\left(\frac{5}{3}\right)}}{\Gamma_q\left(\frac{5}{3}+1\right)} \simeq 1,06214213 \\ \kappa &= \frac{\left([1]_q\right)^{-\frac{5}{3}+\frac{1}{3}} \left(\frac{3}{2}-\frac{1}{2}\right)_q^{\left(\frac{5}{3}-\frac{1}{3}\right)}}{\Gamma_q\left(\frac{5}{3}-\frac{1}{3}+1\right)} \simeq 1,02830547. \end{aligned}$$

Condition (15) satisfies

$$\frac{\lambda u_2^*}{\left(1-\kappa u_3^*\right)} = 0,53955237 < 1.$$

It follows from Theorem 14 that problem (15) has at least one solution.

Example 24. *Consider the following problem*

$$\begin{cases} {}^{\rho}\mathcal{D}_{q,a^+}^{\alpha,\eta}\left(x-h\right)(t) = \dfrac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)\left(1+|x(t)|+\left|{}^{\rho}\mathcal{D}_{q,a^+}^{\beta,\eta}(x-h)(t)\right|\right)},\ t\in\left[\frac{1}{2},1\right], \\ \left({}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-1}x\right)(a) = \omega\in\mathbb{C},\ \text{and}\ \left({}^{\rho}\mathcal{D}_{q,a^+}^{\gamma-k}x\right)(a) = 0,\ \text{for}\ k=2,\ 3,\dots,\ n. \end{cases} \tag{23}$$

We set

$$\psi(t,x,y) = \frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)\left(1+|x|+|y|\right)},\ \text{for}\ x,y\in\mathbb{C},$$

where $\alpha=\frac{7}{3}$*,* $q=\eta=\frac{1}{2}$*,* $\beta=\frac{4}{3}$*, and* $\rho=1$*. Then* $\gamma=\frac{8}{3}$*,* $\theta=\frac{5}{3}$*, and* $n=3$*.*

The function ψ is continuous and for any $x,y,\tilde{x},\tilde{y}\in\mathbb{C}$, and $t\in\left[\frac{1}{2},1\right]$, we have

$$\begin{aligned} & \left|\psi(t,x,y)-\psi(t,\tilde{x},\tilde{y})\right| \\ = & \left|\frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)\left(1+|x|+|y|\right)} - \frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(\sqrt{q}+1\right)\left(1+|\tilde{x}|+|\tilde{y}|\right)}\right| \\ = & \frac{1}{\sqrt{q}+1}\left|\frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(1+|x|+|y|\right)} - \frac{e^{-\pi^2\left(t-\frac{1}{2}\right)}}{\left(1+|\tilde{x}|+|\tilde{y}|\right)}\right| \\ \leq & \frac{1}{\sqrt{q}+1}\left(|x-\tilde{x}|+|y-\tilde{y}|\right). \end{aligned}$$

Hence, the hypothesis $(\mathcal{H}_1)$ is satisfied with

$$c_1 = c_2 = \frac{1}{\sqrt{q}+1} = \frac{1}{\sqrt{\frac{1}{2}}+1} \simeq 0,58578648.$$

and

$$\lambda = \frac{\left([1]_q\right)^{-\frac{7}{3}} \left(1-\frac{1}{2}\right)_q^{\left(\frac{7}{3}\right)}}{\Gamma_q\left(\frac{7}{3}+1\right)} \simeq 0,19842513.$$

$$\kappa = \frac{\left([1]_q\right)^{-\frac{7}{3}+\frac{4}{3}} \left(1-\frac{1}{2}\right)_q^{\left(\frac{7}{3}-\frac{4}{3}\right)}}{\Gamma_q\left(\frac{7}{3}-\frac{4}{3}+1\right)} = 0,5.$$

Now, show that condition (19)

$$\frac{c_1\lambda}{1-\kappa c_2} \simeq 0,16438076 < 1.$$

Is satisfied. It follows from Theorem 15 that problem (22) has a unique solution.

Conclusion and Perspectives

In this work, we have studied the existence, uniqueness, and stability of solutions for a class of nonlinear fractional q-differential equations involving Hilfer-Katugampola q-derivatives of moving orders. By employing the Banach contraction principle and Schauder's fixed-point theorem, we rigorously established the existence of solutions within the Banach space $\mathcal{L}^1_{q,\rho}([a,b],\mathbb{C})$. Moreover, we demonstrated Ulam-Hyers and generalized Ulam-Hyers stability, providing a robust theoretical foundation that ensures the reliability of approximate solutions even in the presence of small perturbations.

The theoretical results were supported by illustrative examples, confirming the applicability of the developed framework. These results contribute to the growing literature on fractional q-calculus and open new avenues for exploring complex dynamical systems characterized by memory and discrete structures.

Future studies could investigate additional aspects of the qualitative behavior of solutions, including asymptotic behavior, oscillatory properties, and long-term dynamics under various structural assumptions. Developing and analyzing efficient numerical methods for solving these nonlinear fractional q-differential equations is a promising direction. These simulations can validate theoretical results and explore analytically intractable solution behaviors.

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