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Traveling Profile Solutions for Parabolic Equations Describing Diffusion Phenomena

Bilal Basti

Abstract This paper aims to investigate and derive new exact solutions for a degenerate parabolic partial differential equation, specifically a nonlinear diffusion equation that is not in divergence form. We propose an approach inspired by the traveling profile method to obtain a general form of self-similar solutions to this equation. The behavior of these solutions depends on certain parameters, which determine whether their existence is global or local in a given time T.

Keywords Traveling profiles · Blow-up · Global · Exact solutions

1 Introduction

Partial differential equations (PDEs) often present significant challenges in theoretical construction, particularly in the case of nonlinear versions. This complexity is evident in equations that describe diffusion phenomena, such as the parabolic PDE known as the nonlinear diffusion equation, which is not in divergence form. This equation is expressed as:

$$\frac{\partial u}{\partial t} = u^m \frac{\partial^2 u}{\partial x^2}, \ m > 0, \tag{1}$$

where $u = u(x, t)$ is a non-negative scalar function of spatial variable $x \in \mathbb{R}$ and time $t \geq 0$.

The nonlinear diffusion equation not in divergence form (1) represents a large class of nonlinear parabolic equations (see [1, 13, 23–25]). These studies have provided results on the existence and uniqueness of global solutions, as well as solutions that blow up in finite time, for certain classes of the function u that satisfy specific sufficient conditions.

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For $m \in (0, 1)$, we consider:

$$\begin{cases} \frac{\partial u}{\partial t} = u^m \frac{\partial^2 u}{\partial x^2}, \ m \in (0, 1) \\ u(x, 0) = u_0(x). \end{cases} \tag{2}$$

This model was studied by Wenshu and Zhengan [25], who proved the existence of a unique solution for any initial condition $u_0 \in C(\mathbb{R}, \mathbb{R}) \cap L^\infty(\mathbb{R}, \mathbb{R})$.

Similarly, Hulshof and Vàzquez [17] studied viscosity solutions of the problem (2). However, they only established the existence of a maximal viscosity solution with nonnegative, continuous, and compactly supported initial data, leaving the uniqueness of viscosity solutions unresolved.

On the other hand, for $m = 1$, the Eq. (1) was investigated by Ferreira et al. [13], on the half-line $x \in (0, \infty)$ with $t \in (0, T)$. Their study particularly focused on the behavior of solutions near $T > 0$, where T represents the maximal existence time for the solution u, which may be finite or infinite.

1. If $T = +\infty$, i.e. $t \in (0, \infty)$, the existence of the solution is global in time.
2. If $T < +\infty$ and

$$\lim_{t \to T} \|u(\cdot, t)\|_{L^\infty} = +\infty,$$

the solution u becomes unbounded in finite time, and we say that it blows up.

In general, for certain PDEs characterized by symmetries [4–8, 10–12, 18–21], exact solutions can be determined through specific finite transformations. These transformations reduce the PDE to an ordinary differential equation (ODE), yielding what are known as "self-similar solutions" [4, 5, 7, 14–16, 19, 24]. Self-similar solutions are paramount in the analysis of PDEs because they provide a unified framework for solving the equations locally and globally. This equivalence underscores their central role in the study of PDEs, as they often reveal fundamental insights into the behavior and properties of the solutions.

Wang and Jingxue [24] thoroughly investigated the existence and uniqueness of a shrinking self-similar solution to the Eq. (1) for $m \geq 1$. The proposed solution was:

$$u(x, t) = \frac{1}{(t+1)^\beta} \omega\left((t+1)^\alpha x^2\right),$$

where ω is a positive function satisfying some properties and

$$\alpha \geq 0, \ \beta = \frac{1+\alpha}{m},$$

are constants selected so that the solutions exist.

Our objective in this work is to study the parabolic PDE (1) and to find exact solutions in the general self-similar form:

$$u(x,t) = c(t)\, f\left(\frac{x-b(t)}{a(t)}\right), \text{ with } a, c \in \mathbb{R}^*, b \in \mathbb{R}.$$

Here $f > 0$ is the "base profile", a, b, and c are functions of time t to be determined.

The approach presented next is inspired by the method known as the "Traveling Profile Method" [2, 7, 8, 10]. This method enables us to obtain many exact solutions for large classes of nonlinear partial differential equations.

2 Traveling Profile Solutions

In this section, we consider $f \in H^2(\mathbb{R},\mathbb{R}) \cap C^2(\mathbb{R},\mathbb{R})$, a nonnegative scalar function referred to as the base profile. The Sobolev space H^2 is defined as follows:

$$H^2(\mathbb{R},\mathbb{R}) = \left\{ f \in L^2(\mathbb{R},\mathbb{R}),\, f' \in L^2(\mathbb{R},\mathbb{R}),\, f'' \in L^2(\mathbb{R},\mathbb{R}) \right\}.$$

Now, let

$$u(x,t) = c(t)\, f\left(\frac{x-b(t)}{a(t)}\right), \text{ with } a, c \in \mathbb{R}^*, b \in \mathbb{R}. \tag{3}$$

be the traveling profile solution for the PDE (1). If we set $\eta = \frac{x-b(t)}{a(t)}$, then $u(x,t) = c(t)\, f(\eta)$ and

$$\frac{\partial u}{\partial t} = \dot{c}(t)\, f - \frac{\dot{a}(t)}{a(t)} c(t)\, \eta f' - \frac{\dot{b}(t)}{a(t)} c(t)\, f'. \tag{4}$$

Similarly, we obtain:

$$\frac{\partial^2 u}{\partial x^2} = \frac{c(t)}{a^2(t)} f''. \tag{5}$$

Substituting (4) and (5) into (1), we get the following equation:

$$\frac{\dot{c}(t)}{c(t)} f - \frac{\dot{a}(t)}{a(t)} \eta f' - \frac{\dot{b}(t)}{a(t)} f' = \frac{c^m(t)}{a^2(t)} f^m f''.$$

This equation involves several unknown parameters, and our objective is to determine the coefficients a, b, and c along with the base profile f.

In principle, the coefficients a, b, and c are functions determined by solving the following minimization problem [8]:

$$\min_{\dot{a},\dot{b},\dot{c}} \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial t} - u^m \frac{\partial^2 u}{\partial x^2} \right|^2 dx.$$

Therefore, we obtain the following three orthogonality equations (see [7, 8])

$$\begin{cases} \left\langle \frac{\partial u}{\partial t} - u^m \frac{\partial^2 u}{\partial x^2}, f \right\rangle = 0, \\ \left\langle \frac{\partial u}{\partial t} - u^m \frac{\partial^2 u}{\partial x^2}, \eta f' \right\rangle = 0, \\ \left\langle \frac{\partial u}{\partial t} - u^m \frac{\partial^2 u}{\partial x^2}, f' \right\rangle = 0, \end{cases} \tag{6}$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product in the space $L^2(\mathbb{R}, \mathbb{R})$, defined by

$$\langle u , v \rangle = \int_{\mathbb{R}} u(\eta)\, v(\eta)\, d\eta.$$

The PDE (1) is thus transformed into a set of three coupled ODEs:

$$\begin{cases} \frac{\dot{c}}{c} \langle f, f \rangle - \frac{\dot{a}}{a} \langle \eta f', f \rangle - \frac{\dot{b}}{a} \langle f', f \rangle = \frac{c^m}{a^2} \langle f^m f'', f \rangle \\ \frac{\dot{c}}{c} \langle f, \eta f' \rangle - \frac{\dot{a}}{a} \langle \eta f', \eta f' \rangle - \frac{\dot{b}}{a} \langle f', \eta f' \rangle = \frac{c^m}{a^2} \langle f^m f'', \eta f' \rangle \\ \frac{\dot{c}}{c} \langle f, f' \rangle - \frac{\dot{a}}{a} \langle \eta f', f' \rangle - \frac{\dot{b}}{a} \langle f', f' \rangle = \frac{c^m}{a^2} \langle f^m f'', f' \rangle . \end{cases} \tag{7}$$

We denote by $(z)_+$ the positive part of z, which is z if $z > 0$ and zero otherwise.

Theorem 1 *Let a, b, and c be three real functions of $t \geq 0$, which satisfy:*

$$a(0) = 1,\ b(0) = 0 \text{ and } c(0) = 1. \tag{8}$$

Then, for $f \in H^2(\mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}, \mathbb{R})$, the Eq. (1) admits an exact solution in the form (3), if the base profile f is a solution of following differential equation:

$$f^m f'' = \alpha f + \beta \eta f' + \gamma f',\ \text{with } \alpha, \beta, \gamma \in \mathbb{R}. \tag{9}$$

We then consider the following cases:

- *If $\alpha m + 2\beta \neq 0$, the coefficients a, b, and c are given by:*

$$\begin{cases} a(t) = (1 - (\alpha m + 2\beta)\, t)_+^{\frac{\beta}{\alpha m + 2\beta}} \\ b(t) = \frac{\gamma}{\beta} (1 - (\alpha m + 2\beta)\, t)_+^{\frac{\beta}{\alpha m + 2\beta}} - \frac{\gamma}{\beta} \\ c(t) = (1 - (\alpha m + 2\beta)\, t)_+^{-\frac{\alpha}{\alpha m + 2\beta}} \end{cases} ,\ 0 \leq t < T. \tag{10}$$

The moment T represents the maximal existence value of the coefficients a, b, and c, such that

$$\begin{cases} T = \frac{1}{\alpha m + 2\beta}, & \text{if } \alpha m + 2\beta > 0, \\ T = +\infty, & \text{if } \alpha m + 2\beta < 0. \end{cases}$$

- *If $\alpha m + 2\beta = 0$, the coefficients a, b and c, in this case, are given by:*

$$\begin{cases} a(t) = e^{-\beta t} \\ b(t) = \frac{\gamma}{\beta} e^{-\beta t} - \frac{\gamma}{\beta} \\ c(t) = e^{\alpha t} \end{cases}, \forall t \geq 0. \tag{11}$$

Proof The proof of this theorem is based on the results shown presented in [8] for an evolution differential operator. In the following, we recall these results in a special case where the differential operator $A_x u = \frac{\partial^2 u}{\partial x^2}$ is multiplied by the nonlinear term u^m for $m > 0$. Let

$$V_t = \{f, \eta f', f'\},$$

be a subspace of $L^2(\mathbb{R}, \mathbb{R})$ generated by the associated function f at the moment t. This implies that the functions f, $\eta f'$, and f' are linearly independent and their inner products are non-zero.

From (6), we deduce that the equation $\frac{\partial u}{\partial t} - u^m \frac{\partial^2 u}{\partial x^2}$ is orthogonal to the subspace V_t. In particular, we have $\frac{\partial u}{\partial t} \in V_t$, thus

$$\left\langle \frac{\partial u}{\partial t} - u^m \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \right\rangle = 0.$$

Therefore, if $u^m \frac{\partial^2 u}{\partial x^2}$ also belongs to V_t, the method provides us a weak exact solution in the form (3).

According to the principle of the method's evaluation, if $u^m \frac{\partial^2 u}{\partial x^2} = \frac{c^{m+1}}{a^2} f^m f''$ belongs to V_t, then

$$u(x,t) = c(t) f\left(\frac{x - b(t)}{a(t)}\right),$$

is an exact solution for the Eq. (1). In this case, the term $f^m f''$ can be expressed as a linear combination of the functions f, $\eta f'$, and f'. That is,

$$f^m f'' = \alpha f + \beta \eta f' + \gamma f', \text{ with } \alpha, \beta, \gamma \in \mathbb{R}.$$

The coefficients a, b, and c are obtained as follows.

If we replace $f^m f''$ by the combination $\alpha f + \beta \eta f' + \gamma f'$ in the system (7), we obtain:

$$MX = \frac{c^m}{a^2} MY, \tag{12}$$

with

$$M = \begin{pmatrix} \langle f, f \rangle & \langle \eta f', f \rangle & \langle f', f \rangle \\ \langle f, \eta f' \rangle & \langle \eta f', \eta f' \rangle & \langle f', \eta f' \rangle \\ \langle f, f' \rangle & \langle \eta f', f' \rangle & \langle f', f' \rangle \end{pmatrix}, X = \begin{pmatrix} \frac{\dot{c}}{c} \\ -\frac{\dot{a}}{a} \\ -\frac{\dot{b}}{a} \end{pmatrix}, Y = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

The matrix M in system (12) is symmetric and invertible, thus it becomes:

$$X = \frac{c^m}{a^2} M^{-1} M Y = \frac{c^m}{a^2} Y,$$

which can be written as:

$$\begin{cases} \dot{a} = -\beta \frac{c^m}{a}, \\ \dot{b} = -\gamma \frac{c^m}{a}, \\ \dot{c} = \alpha \frac{c^{m+1}}{a^2}. \end{cases} \tag{13}$$

Then $\frac{\dot{a}}{a} = -\frac{\beta}{\alpha} \frac{\dot{c}}{c}$, and we deduce:

$$a(t) = K c^{-\frac{\beta}{\alpha}}(t), \, K \in \mathbb{R}. \tag{14}$$

Assuming the conditions (8) are satisfied, thus $K = 1$. Substituting (14) in (13), we obtain:

$$c^{-\frac{\alpha m + 2\beta}{\alpha} - 1} dc = \alpha dt. \tag{15}$$

If $\alpha m + 2\beta \neq 0$, the solution of (15) is:

$$c(t) = (1 - (\alpha m + 2\beta) t)_+^{-\frac{\alpha}{\alpha m + 2\beta}}.$$

Similarly, we get:

$$\begin{cases} a(t) = (1 - (\alpha m + 2\beta) t)_+^{\frac{\beta}{\alpha m + 2\beta}} \\ b(t) = \frac{\gamma}{\beta} (1 - (\alpha m + 2\beta) t)_+^{\frac{\beta}{\alpha m + 2\beta}} - \frac{\gamma}{\beta}. \end{cases}$$

We deduce that the coefficients a, b, and c are defined globally if $\alpha m + 2\beta < 0$, and the coefficients are maximal functions if $\alpha m + 2\beta > 0$, and well defined if and only if

$$0 < t < T = \frac{1}{\alpha m + 2\beta}.$$

If $\alpha m + 2\beta = 0$, the coefficients a, b, and c are defined globally. In this case, we obtain:

$$\begin{cases} a(t) = e^{-\beta t} \\ b(t) = \frac{\gamma}{\beta} e^{-\beta t} - \frac{\gamma}{\beta} \\ c(t) = e^{\alpha t} \end{cases}, \forall t \geq 0.$$

We observe from this theorem that there are two distinct time behaviors of the coefficients a, b, and c. These behaviors depend on the similarity parameters α, β and γ.

3 Global Existence and Blow-up of Solutions

In this section, we provide the sufficient conditions on the parameters α, β, and γ to determine whether the solutions under the traveling profile form are global or exhibit blow-up behavior.

Theorem 2 *Let a, b, and c be three real functions of $t \geq 0$, which satisfy the condition (8), and let $f \in H^2(\mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}, \mathbb{R})$ be a solution of the following differential equation:*

$$f^m f'' = \alpha f + \beta \eta f' + \gamma f'.$$

If

$$\alpha m + 2\beta \leq 0,$$

then the Eq. (1) admits a global solution in time under the traveling profile form, defined for any $t \geq 0$. Moreover, if the profile f is a bounded function and $\alpha < 0$, we have:

$$\lim_{t \to +\infty} u(x, t) = 0, \text{ for all } x \in \mathbb{R}.$$

Conversely, if

$$\alpha m + 2\beta > 0 \text{ and } \alpha > 0,$$

the Eq. (1) admits a solution under the traveling profile form, which blows up in finite time. The solution is defined for all $t \in [0, T)$, the moment T represents the blow-up time of the solution such that:

$$\text{for all } x \in \mathbb{R}, \lim_{t \to T^-} u(x, t) = +\infty, \text{ with } T = \frac{1}{\alpha m + 2\beta} > 0.$$

Proof We have already proved that the traveling profile solution (3) is an exact solution of equation (1), provided that the base profile f is a solution of differential equation (9). Assuming the conditions (8) are satisfied. If $\alpha m + 2\beta < 0$, then the coefficients a, b, and c are given by:

$$\begin{cases} a(t) = (1 - (\alpha m + 2\beta) t)_+^{\frac{\beta}{\alpha m + 2\beta}} \\ b(t) = \frac{\gamma}{\beta} (1 - (\alpha m + 2\beta) t)_+^{\frac{\beta}{\alpha m + 2\beta}} - \frac{\gamma}{\beta} \\ c(t) = (1 - (\alpha m + 2\beta) t)_+^{-\frac{\alpha}{\alpha m + 2\beta}} \end{cases}, \forall t \geq 0$$

and if $\alpha m + 2\beta = 0$, the coefficients a, b, and c are given by:

$$\begin{cases} a(t) = e^{-\beta t} \\ b(t) = \frac{\gamma}{\beta} e^{-\beta t} - \frac{\gamma}{\beta} \\ c(t) = e^{\alpha t} \end{cases}, \forall t \geq 0.$$

In each case, the coefficients a, b, and c are defined globally in time. Now, if $\alpha < 0$, we have $\lim_{t\to+\infty} c(t) = 0$, thus:

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} c(t)\, f\left(\frac{x-b(t)}{a(t)}\right) = 0,$$

if and only if the profile f is a bounded function for all $x \in \mathbb{R}$.

Now, we recall that the solution blows up in finite time if there exists $T < +\infty$, which we call the blow-up time, such that the solution is well-defined for all $0 \le t < T$, while

$$\sup_{x\in\mathbb{R}} |u(x,t)| \to +\infty, \text{ when } t \to T^-.$$

In the case $\alpha m + 2\beta > 0$, $c(t)$ is well-defined if and only if

$$0 \le t < T = \frac{1}{\alpha m + 2\beta}.$$

If $\alpha > 0$, the value T represents the blow-up time of the solution, thus $\lim_{t\to T^-} c(t) = +\infty$, and

$$\lim_{t\to T^-} u(x,t) = \lim_{t\to T^-} c(t)\, f\left(\frac{x-b(t)}{a(t)}\right) = +\infty.$$

4 New Explicit Solutions for the Nonlinear Diffusion Equation

Now, we present some new explicit solutions on the traveling profile form for Eq. (1), where the profile f is an integrable function on $\mathbb{R}$, with

$$\int_{\mathbb{R}} f(\xi)\, d\xi = M, \text{ with } M > 0.$$

Therefore, for $0 < m \neq 1$ and

$$\int_{\mathbb{R}} u(s,t)\, ds = c^m(t),$$

we can explicitly find a new exact solution for the PDE (1). In fact,

$$\int_{\mathbb{R}} u(s,t)\, ds = \int_{\mathbb{R}} c(t)\, f\left(\frac{s-b(t)}{a(t)}\right) ds = a(t)\, c(t) \int_{\mathbb{R}} f(\xi)\, d\xi = c^m(t),$$

this implies that

$$\frac{c^{m-1}(t)}{a(t)} = M. \tag{16}$$

According to the formulas of the functions a and c in (10) and (11), the equality (16) implies that:

$$\beta = \alpha(1-m). \tag{17}$$

In this case, the Eq. (9) becomes:

$$f'' = \left(\frac{1}{1-m}(\beta\eta+\gamma) f^{1-m} + k_0\right)', \text{ where } k_0 \in \mathbb{R}.$$

If we assume $f'(0) = \frac{\gamma}{1-m} f^{1-m}(0)$, after integration, we get:

$$f^{m-1} f' = \frac{1}{1-m}(\beta\eta+\gamma),$$

or

$$\frac{1}{m} f^m = \frac{1}{1-m}\left(\frac{\beta}{2}\eta^2 + \gamma\eta\right) + k_1,$$

for some $k_1 \in \mathbb{R}$. Finally the solution is written as follows:

$$f(\eta) = \left(\frac{\beta m}{2(1-m)}\eta^2 + \frac{\gamma m}{1-m}\eta + k\right)_+^{\frac{1}{m}}, \text{ with } k \in \mathbb{R}.$$

Now, we determine the coefficients a, b, and c. If $\alpha m + 2\beta = \alpha(2-m) \neq 0$, i.e., $m \neq 2$, then

$$\begin{cases} a(t) = (1-\alpha(2-m)t)_+^{\frac{m-1}{m-2}} \\ b(t) = \frac{\gamma}{\beta}(1-\alpha(2-m)t)_+^{\frac{m-1}{m-2}} - \frac{\gamma}{\beta} \\ c(t) = (1-\alpha(2-m)t)_+^{\frac{1}{m-2}} \end{cases}, 0 < t < T.$$

where T represents the maximal existence value:

$$\begin{cases} T = \frac{1}{\alpha(2-m)}, & \text{if } \alpha(2-m) > 0, \\ T = +\infty, & \text{if } \alpha(2-m) < 0. \end{cases}$$

We obtain the exact solution of (1) as follows:

$$u(x,t) = c(t)\left(\frac{\beta m}{2(1-m)}\left(\frac{x-b(t)}{a(t)}\right)^2 + \frac{\gamma m}{1-m}\left(\frac{x-b(t)}{a(t)}\right) + k\right)_+^{\frac{1}{m}}. \tag{18}$$

If $m = 2$, the coefficients a, b, and c are given by (11), and in this case, we obtain:

$$u(x,t) = e^{\alpha t}\left(k - \frac{1}{\beta}\left[(\beta x + \gamma)e^{\beta t} - \gamma\right]^2 - 2\gamma\left[(\beta x + \gamma)e^{\beta t} - \gamma\right]\right)_+^{\frac{1}{2}}.$$

In each case, $k \in \mathbb{R}$ is an arbitrary constant.

For the coefficients a, b, and c that satisfy the parameters of the classical self-similar case, i.e.,

$$c(t) = t^{\alpha}, a(t) = t^{-\beta},\ b(t) = 0,$$

the solution of the equation (1) is written as follows:

$$u(x,t) = t^{\alpha} f(\eta),\ \text{with } \eta = xt^{\beta},$$

where α and β are exponents that satisfy the similarity condition [18–21]

$$\alpha m + 2\beta + 1 = 0, \tag{19}$$

and the function f is the self-similar solution determined by the solution of the following differential equation:

$$f^m f'' = \alpha f + \beta\eta f'.$$

If $\beta = \alpha(1-m)$, the similarity condition (19) gives us:

$$\alpha = \frac{1}{m-2} \text{ and } \beta = \frac{1-m}{m-2}.$$

For $m \in (0,1)$ we get $\alpha, \beta < 0$. Thus, the solution (18) for $\gamma = 0$ and $k > 0$, is written as follows:

$$\begin{aligned}\mathcal{U}(x,t) &= t^{\alpha}\left(k + \frac{\beta m}{2(1-m)}\left(\frac{x}{t^{-\beta}}\right)^2\right)_+^{\frac{1}{m}} \\ &= \left[\left(\frac{1}{1-m}\right)^{\frac{1-m}{m}} t^{\alpha(1-m)}\left(k(1-m) + \frac{\beta m}{2}x^2 t^{2\beta}\right)_+^{\frac{1-m}{m}}\right]^{\frac{1}{1-m}}.\end{aligned}$$

If we set $m \in (0,1)$,

$$p = \frac{1}{1-m} > 1, \alpha_0 = -\frac{\alpha}{p},\ \text{and } \beta_0 = -\beta,$$

then the solution of the equation (1) in the classical self-similar case is written as follows:

$$\mathcal{U}(x,t) = \left[p^{\frac{1}{p-1}} B_C(x,t;\alpha_0,\beta_0,A) \right]^p,$$

where

$$B_C(x,t;\alpha_0,\beta_0,A) = t^{-\alpha_0} \left(C - A \frac{x^2}{t^{2\beta_0}} \right)_+^{\frac{1}{p-1}}, \tag{20}$$

and $C = \frac{k}{p} > 0$ is a free constant. The parameters α_0, β_0, and A have precise values:

$$\alpha_0 = \beta_0 = \frac{1}{p+1}, \text{ and } A = \frac{\beta_0 (p-1)}{2p}. \tag{21}$$

After the implicit change of variables:

$$v = p^{\frac{1}{p-1}} u^{\frac{1}{p}}, \text{ with } p = \frac{1}{1-m} \text{ where } m \in (0,1),$$

the Eq. (1) can be written in the divergence form of the porous media equation (see [3, 17, 22])

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v^p}{\partial x^2}, \; p > 1. \tag{22}$$

The porous media equation (22) admits the properties of similarity. There are several known fundamental families of self-similar solutions, perhaps the most important one is formed by the Barenblatt solutions, discovered independently by Barenblatt in [3] and by Zeldovich and Kompaneets in [26], which are written under the form (20)–(21) for $p > 1$ and gived by:

$$B_C = t^{-\frac{1}{p+1}} \left(C - \frac{p-1}{2p(p+1)} x^2 t^{-\frac{2}{p+1}} \right)_+^{\frac{1}{p-1}},$$

where $C > 0$ is a free constant.

5 Conclusion

In this work, we have discovered new solutions to a nonlinear diffusion equation that is not in divergence form. The behavior of these solutions depends on certain parameters that satisfy specific conditions, determining whether the existence is global or local in time T. The method employed is inspired by the traveling profile method. This method is based on the decomposition of the differential operator in a subspace of L^2 which is generated by functions associated with the base profile f. We have generalized the families of self-similar solutions of the porous media equation, which are represented by the Barenblatt solutions.

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