Transitivities of ternary relations

Norelhouda Bakri
E-mail: norelhouda.bakri@gmail.com
Laboratory of pure and applied mathematics, University of M'sila
This is a joint work with:
L. Zedam — University of M'sila, B. De Baets — University of Gent

Abstract

In this work, we focus on one of the most important properties of relations, it is the transitivity property. it is well known that the transitivity property is characterized by the composition operation. We propose to follow this direction and discuss six types of transitivity of a ternary relation.

Ternary relations

A ternary relation T on a set X is a subset of X^3. Three special ternary relations on a set X are the empty relation $O_{X^3} = \emptyset$, the ternary identity relation $I_{X^3} = \{(x,x,x) \mid x \in X\}$ and the universal ternary relation X^3. A ternary relation T on a set X is called:

(i) *Reflexive*, if, for any $x \in X$, it holds that $(x,x,x) \in T$;

(ii) *Symmetric*, if, for any $x,y,z \in X$, it holds that $(x,y,z) \in T$ implies that $(z,y,x) \in T$.

Inclusion, intersection and union of ternary relations on X are defined through the corresponding notions for subsets of X^3.

Transitivities of a ternary relation

Where the definition of reflexivity and symmetry are quite standard, various notions of transitivity of ternary relations, have been proposed in literature.

(i) $\diamond_1$-transitive, if for any $x,y,z,t \in T$, it holds that $(t,x,y) \in T$ and $(t,y,z) \in T$ imply that $(x,y,z) \in T$;

(ii) $\diamond_2$-transitive, if for any $x,y,z,t \in T$, it holds that $(x,y,t) \in T$ and $(x,t,z) \in T$ imply that $(x,y,z) \in T$;

(iii) $\diamond_3$-transitive, if for any $x,y,z,t \in T$ and $y \neq t$, it holds that $(x,y,t) \in T$ and $(y,t,z) \in T$ imply that $(x,y,z) \in T$.

(iv) $\diamond_4$-transitive, if for any $x,y,z,t \in T$, it holds that $(x,y,t) \in T$ and $(y,z,t) \in T$ imply that $(x,y,z) \in T$;

(v) $\diamond_5$-transitive, if for any $x,y,z,t \in T$, it holds that $(x,t,z) \in T$ and $(t,y,z) \in T$ imply that $(x,y,z) \in T$;

(vi) $\diamond_6$-transitive, if for any $x,y,z,t \in T$, it holds that $(x,y,t) \in T$ and $(t,y,z) \in T$ imply that $(x,y,z) \in T$.

It is clear that the $\diamond_1$-transitivity and $\diamond_3$-transitivity of T implies the $\diamond_2$-transitivity of T.

Compositions of ternary relations

It is well known that the composition is the most important operation that allows to combine relations. There exist many different notions that are characterized by the composition of relations. For instance, the notion of transitivity of binary relations is characterized by the composition operation. In this section, we propose to follow this direction and introduce six types of composition of ternary relations based on the above transitivities as follows:

Definition

Let T and S be two ternary relations on a set X. The $\diamond_i$-compositions of T and S, where $i \in \{1,\ldots,6\}$, are defined as

(i) $T \diamond_1 S := \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,y,z) \in S)\}$;

(ii) $T \diamond_2 S := \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (x,t,z) \in S)\}$;

(iii) $T \diamond_3 S := \{(x,y,z) \in X^3 \mid (\exists t \in X)$ and $(y \neq t)((x,y,t) \in T \wedge (y,t,z) \in S)\}$;

(iv) $T \diamond_4 S := \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (y,z,t) \in S)\}$;

(v) $T \diamond_5 S := \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t,z) \in T \wedge (t,y,z) \in S)\}$;

(vi) $T \diamond_6 S := \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,y,z) \in S)\}$.

The (n,i)-th power $T^{n,\diamond_i}$ of a ternary relation T on a set X is recursively defined as

$$T^{1,\diamond_i} = T \quad \text{and} \quad T^{n+1,\diamond_i} = T^{n,\diamond_i} \diamond_i T,$$

for any $n \in^*$ and $i \in \{1,...,6\}$.

The following proposition shows that the powers of a $\diamond_i$-transitive ternary relation are subsets of this ternary relation.

Proposition

Let T be a ternary relation on a set X. The following equivalence holds:

T is $\diamond_i$-transitive if and only if $T \diamond_i T \subseteq T$, for any $i \in \{1,...,6\}$.

The following proposition shows that for any $i \in \{1,\ldots,6\}$, the $\diamond_i$-transitivity is preserved under intersection, for any $i \in \{1,\ldots,6\}$.

Proposition

Let $(T_j)_{j\in I}$ be a family of ternary relations on a set X. For any $i \in \{1,\ldots,6\}$, it holds that the $\diamond_i$-transitivity of $(T_j)_{j\in I}$ implies the $\diamond_i$-transitivity of $\bigcap_{j\in I} T_j$.

Closures of a ternary relation

If P is a property which a ternary relation T on X may have or fail to have, the $P-$closure of T is defined to be the least inclusive relation S containing T and possessing P.

Definition

If P is a property which a ternary relation T on X may have or fail to have, S is the P-closure of T, written $S = P^{\mathrm{cl}}(T)$, if and only if S satisfies all of

(i) S has property P;

(ii) $T \subseteq S$;

(iii) If $T \subseteq R$ and R has property P, then $S \subseteq R$.

Transitive closures of a ternary relation

One of the most important, probably the most important closures, is the transitive closure. From a practical point of view, the important question arises whether it is possible to find (if it exists) the smallest transitive ternary relation dominating a given ternary relation. Recall that, for any $i \in \{2,5,6\}$, the $\diamond_i$-transitive closures always exist, while the other closures does not exist in general. As in the binary case, the transitive closure of a ternary relation is written with the union of powers. In the following proposition, we introduce the transitive closures of a given ternary relation.

Proposition

Let T be a ternary relation on a set X. It holds that, for any $i \in \{2,5,6\}$, the $\diamond_i$-transitive closure of T is

$$\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T) = \bigcup_{n\geq 1} T^{n,\diamond_i}.$$

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