Some properties of ternary relations and their closures

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Motivation

Relations come in many flavors, such as binary or ternary, crisp or fuzzy, et cetera. Surprisingly, in contrast to binary relations, ternary and, more generally, n-ary relations, have received far less attention. However, in recent years, the interest in ternary relations is on the rise [2, 3].

Ternary relations can display various interesting properties, such as reflexivity, symmetry, cyclicity and transitivity, some of which do not exist in the binary case (such as cyclicity) or come in a multitude of variations in the ternary case (such as transitivity).

In case a binary relation R does not possess a desired property P, the question arises whether it is possible to find (if it exists) the smallest binary relation including R and possessing property P, which is called its P-closure. The main aim is to derive results similar to those of binary relations [1] for the setting of ternary relations.

Outline

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Ternary relations obtained by permutation

Definition

As a binary relation R on a set X is a subset of X^2, a ternary (or triadic) relation T on X is a subset of X^3.

A permutation σ of a 3-element set $U = \{u, v, w\}$ is a bijection from U to itself. The six permutations of U are given by

$$\sigma_0(u,v,w) = (u,v,w), \quad \sigma_1(u,v,w) = (u,w,v), \quad \sigma_2(u,v,w) = (v,u,w),$$

$$\sigma_3(u,v,w) = (v,w,u), \quad \sigma_4(u,v,w) = (w,u,v), \quad \sigma_5(u,v,w) = (w,v,u).$$

The ternary relation T^σ on X is defined as

$$T^\sigma = \{\sigma(x,y,z) \in X^3 \mid (x,y,z) \in T\}.$$

It is clear that $T^{\sigma_0} = T$ and $T^{\sigma_5} = T^t$.

Also, we have the following terminology and notations.

Definition

Let T be a ternary relation on a set X.

(i) The right-converse of T is the ternary relation $T^{\dashv}$ on X defined as $T^{\dashv} = T^{\sigma_1}$;

(ii) The left-converse of T is the ternary relation $T^{\vdash}$ on X defined as $T^{\vdash} = T^{\sigma_2}$;

(iii) The right-rotation of T is the ternary relation T^{+} on X defined as $T^{+} = T^{\sigma_3}$;

(iv) The left-rotation of T is the ternary relation T^{-} on X defined as $T^{-} = T^{\sigma_4}$.

Properties of ternary relations

A ternary relation T on X is called:

(i) *reflexive*, if, for any $x \in X$, it holds that $(x, x, x) \in T$;

(ii) *symmetric*, if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies $(z, y, x) \in T$;

(iii) *strongly symmetric*, if $T = T^{\sigma_i}$, for any $i \in \{1, .., 5\}$;

(iv) *cyclic*, if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies $(y, z, x) \in T$;

Transitivity of a ternary relation

Where the above definitions of reflexivity and symmetry are quite standard, there is much less agreement in literature on the definition of transitivity of a ternary relation. Indeed, various alternative definitions, each with its own motivation, have been proposed. The ones we have selected below are those that are closest in spirit to the definition of transitivity of a binary relation:

(i) $\circ_1$-*transitive*, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, t) \in T$ and $(s, t, z) \in T$ imply that $(x, y, z) \in T$;

(ii) $\circ_2$-*transitive*, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ imply that $(x, t, z) \in T$;

(iii) $\circ_3$-*transitive*, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ imply that $(x, y, z) \in T$;

(iv) $\circ_4$-*transitive*, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ imply that $(x, y, t) \in T$;

(v) $\circ_5$-*transitive*, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ imply that $(y, t, z) \in T$;

(vi) $\circ_6$-*transitive*, if, for any $x, y, z, s, t \in X$, it holds that $(x, s, t) \in T$ and $(s, y, z) \in T$ imply that $(x, y, z) \in T$.

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P-closures

Definition

If P is a property which a ternary relation T on a set X may have or fail to have, then a ternary relation S is the P-closure of T, written $S = P^{\mathrm{cl}}(T)$, if and only if S satisfies all of

(i) S possesses property P;

(ii) $T \subseteq S$;

(iii) If $T \subseteq T'$ and T' possesses property P, then $S \subseteq T'$.

It is clear that a P-closure, if it exists, must be unique.

When a P-closure exists?

For many properties P, a P-closure exists for some ternary relations but not for others. The interesting properties P are those for which every ternary relation has a P-closure. The following theorem states the conditions for this to occur.

Theorem

A P-closure exists for all ternary relations T on a set X if and only if

(i) *The universal relation X^3 possesses P;*

(ii) *The intersection of every (non-empty) family of ternary relations, each of which possesses P, also possesses P.*

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Closures for some properties of ternary relations

Bandler and Kohout [1] studied the closures for some special properties of binary relations, such as the symmetric closure, transitive closure, preorder closure and equivalence closure. In the following theorem, we follow the same direction and express the closures of a ternary relation for the reflexivity, symmetry, strong symmetry and cyclicity properties. Note that these properties are all preserved under intersection and that the universal relation X^3 possesses all of them. Hence, the closures for these properties always exist.

Theorem

Let T be a ternary relation on a set X. It holds that:

(i) *The reflexive closure is* $\mathrm{Ref}^{\mathrm{cl}}(T) = T \cup I_{X^3}$, *such that* $I_{X^3} = \{(x,x,x) \in X^3 \mid x \in X\}$;

(ii) *The symmetric closure is* $\mathrm{Sym}^{\mathrm{cl}}(T) = T \cup T^t$;

(iii) *The strongly symmetric closure is* $\mathrm{SSym}^{\mathrm{cl}}(T) = \bigcup_{i=0}^{5} T^{\sigma_i}$;

(iv) *The cyclic closure is* $\mathrm{Cyc}^{\mathrm{cl}}(T) = T \cup T^+ \cup T^-$.

Transitive closures of a ternary relation

Note that, for any $i \in \{1, \ldots, 6\}$, the $\circ_i$-transitive closure always exists, as the universal relation X^3 is $\circ_i$-transitive and intersection preserves $\circ_i$-transitivity. As in the binary case, the transitive closure of a ternary relation can be written as the union of powers. However, this only holds for the associative compositions, which are $\circ_i$, for $i \in \{1, 2, 5, 6\}$.

Theorem

Let T be a ternary relation on a set X. It holds that the $\circ_i$-transitive closure of T is $\mathrm{Tr}^{\mathrm{cl}}_{\circ_i}(T) = \bigcup_{n \geq 1} T^{n,\circ_i}$, *for any* $i \in \{1, 2, 5, 6\}$.

Example

Example

Let T be the ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:

$$T = \{(x_1, x_1, x_2), (x_2, x_4, x_1), (x_3, x_2, x_3)\}.$$

One easily computes the $\circ_1$-powers of T:

$$T^{2,\circ_1} = \{(x_1, x_1, x_3), (x_2, x_4, x_2)\},$$
$$T^{3,\circ_1} = \{(x_2, x_4, x_3)\},$$
$$T^{4,\circ_1} = \emptyset.$$

Thus, $\mathrm{Tr}^{\mathrm{cl}}_{\circ_1}(T) = \{(x_1, x_1, x_2), (x_1, x_1, x_3), (x_2, x_4, x_1), (x_2, x_4, x_2),$

$$(x_2, x_4, x_3), (x_3, x_2, x_3)\}.$$

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Conclusions

In this note:

- Discussed some properties of ternary relations.
- Introduced several types of transitivity of ternary relations.
- Expressed the closures for some properties of ternary relations.
- Characterized the transitive closures of a ternary relation.

References

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Thank you for your attention!