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Traveling Wave Solutions for an Initial Value Problem of Generalized Fractional Nonlinear Korteweg–de Vries Equations

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ABSTRACT

This paper comprehensively investigates the existence, uniqueness, and stability of traveling wave solutions for an initial value problem of generalized nonlinear Korteweg–de Vries equations of fractional order. We apply the Banach contraction principle and Schauder's fixed-point theorem to establish the existence and uniqueness of solutions. Furthermore, we examine the stability of the solutions using Ulam–Hyers theorems. Two well-detailed examples and one explicit solution are provided to illustrate the practical applicability and validity of our theoretical results. The solution's parameter conditions detail the amplitude, energy, wavelength, frequency, and propagation characteristics of waves. These parameters capture the intricate balance between nonlinear and dispersive effects that shape the wave phenomena modeled by the Korteweg–de Vries equation. The solution indicates that as the amplitude parameter increases, wave height also rises due to a higher wave number (shorter wavelength) and nonlinear amplification, resulting in more waves within a given interval. The amplitude further increases with the wave number due to higher energy levels associated with shorter wavelengths.

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1 | Introduction

Nonlinear fractional partial differential equations (PDEs) have been used to model several evolutionary phenomena in different scientific areas [1–4]. The Korteweg–de Vries (KdV) equation has been a cornerstone in studying nonlinear wave phenomena. Its applications extend beyond shallow water waves [5], including ion-acoustic waves in plasmas [6], internal waves in stratified fluids [7–9], and even lattice waves in crystal structures [10]. It was first introduced in 1895 by Korteweg and de Vries to model the propagation of long waves in shallow channels [11].

The generalized nonlinear fractional Korteweg–de Vries equation (GNFKdVE) introduces fractional derivatives to better capture the memory and hereditary properties of various materials and media. For this consideration, we propose the following initial value problem (IVP):

$$\begin{cases} \frac{\partial \phi}{\partial t} + \frac{\partial^{\mu} \phi}{\partial p^{\mu}} = \mathcal{F}\left(p, t, \phi, \frac{\partial \phi}{\partial p}\right), & (p,t) \in \mathcal{H}, \\ \phi(\delta t, t) = c_0 \exp(\mu t), & c_0 \in \mathbb{C}, \\ \frac{\partial \phi}{\partial p}(\delta t, t) = \frac{\partial^2 \phi}{\partial p^2}(\delta t, t) = 0, & \delta \in \mathbb{R}_+, \end{cases} \tag{1}$$

where $\mu \in (2,3]$ and $\frac{\partial^\mu \phi}{\partial p^\mu}$ represents the Caputo fractional derivative of order μ with respect to p, such that

$$\frac{\partial^\mu \phi}{\partial p^\mu} = \begin{cases} \frac{d^3\phi}{dp^3}, & \mu = 3, \\ \mathcal{J}_{\delta t}^{3-\mu} \frac{\partial^3 \phi}{\partial p^3} = \frac{1}{\Gamma(3-\mu)} \int_{\delta t}^{p} (p-q)^{2-\mu} \frac{\partial^3}{\partial q^3} \phi(q,t) dq, & 2 < \mu < 3. \end{cases}$$

In this context, the symbol $\mathcal{J}_*^\mu$ denotes the Riemann–Liouville fractional integral of order μ. Furthermore, $\phi = \phi(p,t)$ represents the wave amplitude, width, or height as a function of position p and time t. Additionally, $\mathcal{F} : \mathcal{H} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a nonlinear function, with

$$\mathcal{H} = \{(p,t) \in \mathbb{R} \times [0,T];\ \delta t \le p \le \lambda\}, \text{ for } T > 0 \text{ and } \lambda \ge \delta T.$$

Our study engages in both analytical and numerical discussions of GNFKdVEs to explore the behavior of their exact solutions, offering essential insights into simulating the complex dynamics of these equations. Recent advancements in computational techniques have further enhanced our ability to solve GNFKdVEs efficiently, providing a deeper understanding of their intricate wave phenomena.

Note that, for $\mathcal{F} \equiv \eta\phi\frac{\partial\phi}{\partial p}$, $\eta \in \mathbb{R}$, and $\mu = 3$, the fractional-order PDE in (1) becomes the standard one-dimensional nonlinear KdV equation [12]. As mentioned above, the equation has several applications, and its versatility and integrability make it a powerful tool in both theoretical and applied physics [13].

The existence, uniqueness, and stability of solutions for GNFKdVEs (1) are investigated in this paper, with a specific emphasis on when these equations are presented in the form of traveling wave:

$$\phi(p,t) = \exp(\mu t) y(p - \delta t), \text{ for } \mu \in (2,3] \text{ and } \delta \in \mathbb{R}_+ \quad (2)$$

The base profile, denoted as y, remains undisclosed a priori and requires identification upon reaching our main results. The traveling wave method enables us to transform the fractional-order PDE (1) into a fractional differential equation (FDE). This concept is thoroughly demonstrated with examples in our paper. This approach (2) shows promise and has the potential to produce new findings for other applications involving fractional-order PDEs.

2 | Necessary Definitions and Preliminaries

We elucidate here the essential definitions derived from fractional calculus theory. Let $\mathbb{E} = [0,\lambda]$ be a finite interval. The space under consideration is $C(\mathbb{E},\mathbb{C})$ the Banach space of all continuous functions from $\mathbb{E}$ to $\mathbb{C}$, characterized by the norm

$$\|y\|_\infty = \sup_{z\in\mathbb{E}} |y(z)|.$$

Definition 1. ([2]). The left-sided (arbitrary) fractional integral of order $\mu > 0$ of a continuous function $y : \mathbb{E} \to \mathbb{C}$ is given by

$$\mathcal{J}_{0^+}^\mu y(z) = \frac{1}{\Gamma(\mu)} \int_0^z (z-\tau)^{\mu-1} y(\tau) d\tau, z \in \mathbb{E}.$$

$\Gamma(\mu) = \int_0^\infty \tau^{\mu-1} \exp(-\tau) d\tau$ is the Euler gamma function.

Definition 2. (Caputo fractional derivative [2]). The left-sided Caputo fractional derivative of order $\mu > 0$ of a function $y : \mathbb{E} \to \mathbb{C}$ is given by

$${}^C\mathcal{D}_{0^+}^\mu y(z) = \begin{cases} \frac{d^m y(z)}{dz^m}, & \text{for } \mu = m \in \mathbb{N}_0, \\ \mathcal{J}_{0^+}^{m-\mu} \frac{d^m y(z)}{dz^m} = \int_0^z \frac{(z-\tau)^{m-\mu-1}}{\Gamma(m-\mu)} \frac{d^m y(\tau)}{d\tau^m} d\tau, & \text{for } m-1 < \mu < m \in \mathbb{N}. \end{cases}$$

Lemma 3. ([2]). *Let $\mu > 0$ and assume that ${}^C\mathcal{D}_{0^+}^\mu y \in C(\mathbb{E},\mathbb{C})$, then*

$$\mathcal{J}_{0^+}^{\mu}\, {}^C\mathcal{D}_{0^+}^\mu y(z) = y(z) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{k!} z^k, \quad m-1 < \mu \le m \in \mathbb{N}^*.$$

Definition 4. Let $\mathcal{F} : \mathcal{H} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be a nonlinear continuous function. The equation

$$\frac{\partial\phi}{\partial t} + \frac{\partial^\mu\phi}{\partial p^\mu} = \mathcal{F}\left(p,t,\phi,\frac{\partial\phi}{\partial p}\right), (p,t) \in \mathcal{H} \quad (3)$$

is Ulam–Hyers stable if there exists a real number $\gamma > 0$ such that for each $\varepsilon > 0$ and each solution $\overline{\phi}$, which is $C^3(\mathcal{H},\mathbb{C})$ in position p and $C^1(\mathcal{H},\mathbb{C})$ in time t, of the inequality

$$\left| \frac{\partial\overline{\phi}}{\partial t} + \frac{\partial^\mu\overline{\phi}}{\partial p^\mu} - \mathcal{F}\left(p,t,\overline{\phi},\frac{\partial\overline{\phi}}{\partial p}\right) \right| \le \varepsilon, \forall (p,t) \in \mathcal{H} \quad (4)$$

there exists a solution ϕ of (3), with

$$\left|\overline{\phi}(p,t) - \phi(p,t)\right| \le \gamma\varepsilon, \forall (p,t) \in \mathcal{H}.$$

Definition 5. The equation (3) is generalized Ulam–Hyers stable if there exists $\mathcal{Q} \in C(\mathbb{R}_+,\mathbb{R}_+)$, with $\mathcal{Q}(0) = 0$, such that for each solution $\overline{\phi}$, which is $C^3(\mathcal{H},\mathbb{C})$ in position p and $C^1(\mathcal{H},\mathbb{C})$ in time t, of the inequality (4), there exists a solution ϕ of (3) with

$$\left|\overline{\phi}(p,t) - \phi(p,t)\right| \le \mathcal{Q}(\varepsilon), \forall (p,t) \in \mathcal{H}.$$

3 | Main Work and Findings

We introduce the following assumptions:

($\aleph$1) The function $\mathcal{F}$ is continuous and exhibits invariance of scale (2). It becomes

$$\mathcal{F}\left(p,t,\phi,\frac{\partial\phi}{\partial p}\right) = \exp(\mu t) f\left(z, y(z), y'(z)\right), \quad (5)$$

with $z = p - \delta t$ and $f : \mathbb{E} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function.

($\aleph$2) There are two real constants $\alpha, \beta > 0$ such that the function f defined in (5) satisfies

$$\left|f(z,y,x) - f\left(z,\overline{y},\overline{x}\right)\right| \le \alpha\left|y - \overline{y}\right| + \beta\left|x - \overline{x}\right|, \forall y,x,\overline{y},\overline{x} \in \mathbb{C}.$$

(ℵ3) Three nonnegative functions $u, v, w \in C(\mathbb{E}, \mathbb{R}_+)$ exist such that

$$|f(z, y, x)| \leq u(z) + v(z)|y| + w(z)|x|, \forall z \in \mathbb{E},$$

for any $y, x \in \mathbb{C}$. Furthermore, we denote

$$u^* = \sup_{z\in\mathbb{E}} u(z),\ \ v^* = \sup_{z\in\mathbb{E}} v(z), \text{ and } w^* = \sup_{z\in\mathbb{E}} w(z).$$

3.1 | Traveling Wave Solutions for GNFKdVEs

First, we derive the problem that the function y in (2) satisfies, which is used to define traveling wave forms.

Theorem 6. *Let $c_0 \in \mathbb{C}$, $\delta \in \mathbb{R}_+$ and $2 < \mu \leq 3$. If (ℵ1) holds, then the traveling wave form (2) reduces IVP of GNFKdVEs (1) to the following FDE:*

$$^C\mathcal{D}^{\mu}_{0^+} y(z) = \psi(z), z \in \mathbb{E} \tag{6}$$

where

$$\psi(z) = f\left(z, y(z), y'(z)\right) + \delta y'(z) - \mu y(z),$$

with the conditions

$$y(0) = c_0 \text{ and } y'(0) = y''(0) = 0 \tag{7}$$

Proof. First, for $z = p - \delta t$, we obtain $z \in \mathbb{E}$. Substituting expression (2) in IVP of GNFKdVEs (1) results the following equalities (check also [14–24]):

$$\frac{\partial \phi}{\partial t} = \exp(\mu t)\big(\mu y(z) - \delta y'(z)\big) \tag{8}$$

On the flip side, for $\tau = q - \delta t$, we get

$$\begin{aligned} \frac{\partial^{\mu}\phi}{\partial p^{\mu}} &= \frac{1}{\Gamma(3-\mu)} \int_{\delta t}^{p} (p-q)^{2-\mu} \frac{\partial^3 \phi(q,t)}{\partial q^3} dq \\ &= \frac{\exp(\mu t)}{\Gamma(3-\mu)} \int_{\delta t}^{p} (p-q)^{2-\mu} \frac{d^3 y(q-\delta t)}{dq^3} dq \\ &= \frac{\exp(\mu t)}{\Gamma(3-\mu)} \int_{0}^{z} (z-\tau)^{2-\mu} \frac{d^3 y(\tau)}{d\tau^3} d\tau \\ &= \exp(\mu t)^C\mathcal{D}^{\mu}_{0^+} y(z). \end{aligned} \tag{9}$$

If we replace (5), (8), and (9) in (1), we obtain

$$^C\mathcal{D}^{\mu}_{0^+} y(z) = \psi(z).$$

From (1), we find

$$\phi(\delta t, t) = y(0)\exp(\mu t),\ \frac{\partial \phi}{\partial p}(\delta t, t) = y'(0)\exp(\mu t),$$
$$\text{and } \frac{\partial^2 \phi}{\partial p^2}(\delta t, t) = y''(0)\exp(\mu t).$$

Consequently

$$y(0) = c_0 \text{ and } y'(0) = y''(0) = 0.$$

That establishes the theorem. □

Lemma 7. *Problem (6)–(7) is equivalent to the integral equation*

$$y(z) = c_0 + \frac{1}{\Gamma(\mu)} \int_0^z (z-\tau)^{\mu-1} \psi(\tau) d\tau, \forall z \in \mathbb{E},$$

where $\psi \in C(\mathbb{E}, \mathbb{C})$ satisfies the functional equation

$$\psi(z) = g\left(z, c_0 + \mathcal{J}^{\mu}_{0^+}\psi(z), \psi(z)\right) - \mu\left(c_0 + \mathcal{J}^{\mu}_{0^+}\psi(z)\right),$$

and $g : \mathbb{E}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous function satisfying

$$g(z, y(z), \psi(z)) = \delta \mathcal{J}^{\mu-1}_{0^+}\psi(z) + f\left(z, y(z), \mathcal{J}^{\mu-1}_{0^+}\psi(z)\right).$$

Proof. Using Theorem 6, and applying $\mathcal{J}^{\mu}_{0^+}$ to both sides of Equation (6), we obtain

$$\mathcal{J}^{\mu}_{0^+}{}^C\mathcal{D}^{\mu}_{0^+} y(z) = \mathcal{J}^{\mu}_{0^+}\psi(z).$$

From Lemma 3, we find

$$\mathcal{J}^{\mu}_{0^+}{}^C\mathcal{D}^{\mu}_{0^+} y(z) = y(z) - y(0) - zy'(0) - \frac{1}{2}z^2 y''(0).$$

Substituting (7) gives us

$$y(z) = c_0 + \mathcal{J}^{\mu}_{0^+}\psi(z) \tag{10}$$

As

$$y'(z) = \frac{d}{dz}\left(c_0 + \mathcal{J}^{\mu}_{0^+}\psi(z)\right) = \mathcal{J}^{\mu-1}_{0^+}\psi(z)$$

and

$$y''(z) = \frac{d^2}{dz^2}\left(c_0 + \mathcal{J}^{\mu}_{0^+}\psi(z)\right) = \mathcal{J}^{\mu-2}_{0^+}\psi(z),$$

then

$$\begin{aligned} \psi(z) &= f\left(z, y(z), y'(z)\right) + \delta y'(z) - \mu y(z) \\ &= g\left(z, c_0 + \mathcal{J}^{\mu}_{0^+}\psi(z), \psi(z)\right) - \mu\left(c_0 + \mathcal{J}^{\mu}_{0^+}\psi(z)\right). \end{aligned}$$

Alternatively, by applying $^C\mathcal{D}^{\mu}_{0^+}$ to both sides of Equation (10) and taking advantage of the linearity of Caputo's derivative, as well as the fact that $^C\mathcal{D}^{\mu}_{0^+} c_0 = 0$, we can derive (6). Furthermore,

$$y(0) = \left(c_0 + \mathcal{J}^{\mu}_{0^+}\psi\right)(0) = c_0,$$
$$y^{(k)}(0) = \mathcal{J}^{\mu-k}_{0^+}\psi(0) = 0, \text{ for each } k = 1, 2.$$

The proof is complete. □

3.2 | Principal Theorems' Proofs

This part investigates the existence and uniqueness of traveling wave solutions for IVP of GNFKdVEs (1).

Theorem 8. *Assume that (ℵ1)–(ℵ3) hold. If we set $\theta = \max\{\beta, w^*\}$ and*

$$\Gamma(\mu) > \frac{\mu(\delta+\theta+\lambda)+v^*\lambda}{\mu\lambda^{1-\mu}} \quad (11)$$

then, there is at least one solution for IVP of GNFKdVEs (1) on $\mathcal{H}$ in the traveling wave form (2).

Proof. Assume the assumption (ℵ1) holds. Using Theorem 6, IVP of GNFKdVEs (1) is reduced to fractional-order's problem (6)–(7).

Our proof begins with a transformation of problem (6)–(7) into the fixed point problem $\mathcal{B}\varphi(z) = \varphi(z)$. Let us define

$$\mathcal{B}\varphi(z) = c_0 + \frac{1}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}\psi(\tau)d\tau \quad (12)$$

where

$$\psi(z) = g(z,\varphi(z),\psi(z)) - \mu\varphi(z),\ z\in\mathbb{E},$$

with

$$g(z,\varphi(z),\psi(z)) = \delta\mathcal{J}_{0^+}^{\mu-1}\psi(z) + f\left(z,\varphi(z),\mathcal{J}_{0^+}^{\mu-1}\psi(z)\right).$$

We observe that when $\psi\in C(\mathbb{E},\mathbb{C})$, then $\mathcal{B}\varphi$ remains continuous, as demonstrated in step (***i.***) of this proof. Consequently, $\mathcal{B}$ belongs to $C(\mathbb{E},\mathbb{C})$ and is characterized by the norm

$$\|\mathcal{B}\varphi\|_\infty = \sup_{z\in\mathbb{E}}|\mathcal{B}\varphi(z)|.$$

Regarding problem (6)–(7) is equivalent to (12), which means that $\mathcal{B}$ includes fixed points that solve the above-mentioned problem.

Also, Schauder's fixed point theorem's assumption (see [25]) is satisfied by $\mathcal{B}$. The following steps prove this.

i. $\mathcal{B}$ is a continuous operator.
Let $(\varphi_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty}\varphi_n = \varphi$ in $C(\mathbb{E},\mathbb{C})$. Then for any $z\in\mathbb{E}$,

$$\left|\mathcal{B}\varphi_n(z) - \mathcal{B}\varphi(z)\right| \le \frac{1}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}\left|\psi_n(\tau)-\psi(\tau)\right|d\tau,$$

where

$$\begin{cases}\psi_n(z) = g\left(z,\varphi_n(z),\psi_n(z)\right) - \mu\varphi_n(z),\\ \psi(z) = g(z,\varphi(z),\psi(z)) - \mu\varphi(z).\end{cases}$$

By applying assumption (ℵ2), we get

$$\begin{aligned}&\left|g\left(z,\varphi_n(z),\psi_n(z)\right) - g(z,\varphi(z),\psi(z))\right|\\ &\le \left|\delta\left(\mathcal{J}_{0^+}^{\mu-1}\psi_n - \mathcal{J}_{0^+}^{\mu-1}\psi\right)(z)\right|\\ &\quad + \left|f\left(z,\varphi_n(z),\mathcal{J}_{0^+}^{\mu-1}\psi(z)\right)\right.\\ &\quad \left.-f\left(z,\varphi(z),\mathcal{J}_{0^+}^{\mu-1}\psi(z)\right)\right|\\ &\le \alpha\left\|\varphi_n-\varphi\right\|_\infty + \frac{\delta+\beta}{\lambda^{1-\mu}\Gamma(\mu)}\left\|\psi_n-\psi\right\|_\infty.\end{aligned}$$

Then,

$$\begin{aligned}&\left|\psi_n(z)-\psi(z)\right|\\ &= \left|\mu\left(\varphi(z)-\varphi_n(z)\right) + g\left(z,\varphi_n(z),\psi_n(z) - g(z,\varphi(z),\psi(z))\right)\right|\\ &\le (\mu+\alpha)\left\|\varphi_n-\varphi\right\|_\infty + \frac{\delta+\beta}{\lambda^{1-\mu}\Gamma(\mu)}\left\|\psi_n-\psi\right\|_\infty.\end{aligned}$$

Thus,

$$\left\|\psi_n-\psi\right\|_\infty \le \frac{(\mu+\alpha)\lambda^{1-\mu}\Gamma(\mu)}{\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta}\left\|\varphi_n-\varphi\right\|_\infty.$$

Since $\varphi_n\to\varphi$, we get $\psi_n\to\psi$ when $n\to\infty$.
Now, let $\xi>0$, be such that for each $z\in\mathbb{E}$, we have

$$\left|\psi_n(z)\right|\le\xi, \left|\psi(z)\right|\le\xi.$$

Then, we have

$$\begin{aligned}\frac{(z-\tau)^{\mu-1}}{\Gamma(\mu)}\left|\psi_n(\tau)-\psi(\tau)\right| &\le \frac{(z-\tau)^{\mu-1}}{\Gamma(\mu)}\left[\left|\psi_n(\tau)\right| + |\psi(\tau)|\right]\\ &\le \frac{2\xi}{\Gamma(\mu)}(z-\tau)^{\mu-1}.\end{aligned}$$

The function $\tau\to\frac{2\xi}{\Gamma(\mu)}(z-\tau)^{\mu-1}$ is integrable on $[0,z]$, for each $z\in\mathbb{E}$. Therefore, the implication of Lebesgue's dominated convergence theorem is

$$\left|\mathcal{B}\varphi_n(z)-\mathcal{B}\varphi(z)\right|\to 0 \text{ as } n\to\infty,$$

and hence,

$$\lim_{n\to\infty}\left\|\mathcal{B}\varphi_n-\mathcal{B}\varphi\right\|_\infty = 0.$$

This indicates the continuity of $\mathcal{B}$.

ii. $\mathcal{B}$ is defined from a bounded, closed and convex subset into itself.
Using (11), we define

$$a \ge \frac{\mu\left|c_0\right|\left(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta\right)+u^*\lambda}{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)-(\mu+v^*)\lambda},$$

and

$$C_a = \left\{\varphi\in C(\mathbb{E},\mathbb{C}) : \|\varphi\|_\infty\le a\right\}.$$

Obviously, C_a is a subset of $C(\mathbb{E},\mathbb{C})$ characterized by being bounded, closed and convex.
Consider the integral operator $\mathcal{B} : C_a\to C(\mathbb{E},\mathbb{C})$ defined by (12). It follows that $\mathcal{B}\left(C_a\right)\subset C_a$.
Indeed, using (ℵ3), we have for each $z\in\mathbb{E}$

$$\begin{aligned}|\psi(z)| &= |g(z,\varphi(z),\psi(z)) - \mu\varphi(z)|\\ &\le u^* + (\mu+v^*)|\varphi(z)| + \frac{\delta+w^*}{\lambda^{1-\mu}\Gamma(\mu)}\|\psi\|_\infty.\end{aligned}$$

Then, we get

$$\|\psi\|_\infty \le \frac{\lambda^{1-\mu}\Gamma(\mu)(u^*+(\mu+v^*))a}{\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta}.$$

Thus,

$$\begin{aligned}|\mathcal{B}\varphi(z)| &\le |c_0| + \frac{1}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}|\psi(\tau)|d\tau\\ &\le |c_0| + \frac{u^*\lambda + (\mu+v^*)\lambda a}{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)}\\ &\le \frac{\frac{\mu|c_0|(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)+u^*\lambda}{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)}\frac{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)}{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)-(\mu+v^*)\lambda}}{\frac{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)}{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)-(\mu+v^*)\lambda}}\\ &\quad + \frac{(\mu+v^*)\lambda a}{\mu(\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta)}\\ &\le a.\end{aligned}$$

Then $\mathcal{B}(C_a) \subset C_a$.

iii. $\mathcal{B}(C_a)$ is an equicontinuous subset.
Let $z_1, z_2 \in \mathbb{E}$, $z_1 < z_2$, and $\varphi \in C_a$. Then,

$$\begin{aligned}&\left|\mathcal{B}\varphi(z_2) - \mathcal{B}\varphi(z_1)\right|\\ &= \frac{1}{\Gamma(\mu)}\left|\int_0^{z_2}(z_2-\tau)^{\mu-1}\psi(\tau)d\tau - \int_0^{z_1}(z_1-\tau)^{\mu-1}\psi(\tau)d\tau\right|\\ &\le \frac{1}{\Gamma(\mu)}\int_0^{z_1}\left|\left((z_2-\tau)^{\mu-1}-(z_1-\tau)^{\mu-1}\right)\psi(\tau)\right|d\tau\\ &\quad + \frac{1}{\Gamma(\mu)}\int_{z_1}^{z_2}(z_2-\tau)^{\mu-1}|\psi(\tau)|d\tau\\ &\le \frac{\lambda^{1-\mu}(u^*+(\mu+v^*)a)}{\lambda^{1-\mu}\Gamma(\mu)-\delta-\theta}\left[\int_0^{z_1}\left|(z_2-\tau)^{\mu-1}-(z_1-\tau)^{\mu-1}\right|d\tau\right.\\ &\quad \left.+\int_{z_1}^{z_2}(z_2-\tau)^{\mu-1}d\tau\right].\end{aligned} \tag{13}$$

We have

$$(z_2-\tau)^{\mu-1} - (z_1-\tau)^{\mu-1} = -\frac{1}{\mu}\frac{d}{d\tau}\left[(z_2-\tau)^{\mu} - (z_1-\tau)^{\mu}\right].$$

Then,

$$\begin{aligned}&\int_0^{z_1}\left|(z_2-\tau)^{\mu-1}-(z_1-\tau)^{\mu-1}\right|d\tau\\ &\le \frac{1}{\mu}\left[(z_2-z_1)^{\mu} + (z_2^{\mu}-z_1^{\mu})\right].\end{aligned}$$

We also have

$$\int_{z_1}^{z_2}(z_2-\tau)^{\mu-1}d\tau = -\frac{1}{\mu}\left[(z_2-\tau)^{\mu}\right]_{z_1}^{z_2} \le \frac{1}{\mu}(z_2-z_1)^{\mu}.$$

Thus, (13) gives us

$$\left|\mathcal{B}\varphi(z_2) - \mathcal{B}\varphi(z_1)\right| \le \frac{2(z_2-z_1)^{\mu} + (z_2^{\mu}-z_1^{\mu})}{\lambda^{1-\mu}\Gamma(\mu+1)-\mu(\delta+\theta)}\left(\lambda^{1-\mu}(u^*+(\mu+v^*)a)\right).$$

The right-hand side of the aforementioned inequality converges to zero as z_1 approaches z_2.

As a consequence of steps ***i.***, ***ii.***, and ***iii.***, and with the aid of the Ascoli-Arzelà theorem, we deduce the continuity of $\mathcal{B} : C_a \to C_a$, its compactness, and its compliance with the conditions required by Schauder's fixed point theorem [25]. Consequently, $\mathcal{B}$ possesses a fixed point that solves problem (6)–(7) on $\mathbb{E}$.

Similarly, we can demonstrate the existence of at least one solution for IVP of GNFKdVEs (1) under the traveling wave form (2). This is achievable if (11) holds for any $(p,t) \in \mathcal{H}$. □

Illustrative example 1. Let us consider the following IVP of GNFKdVEs

$$\begin{cases}\frac{\partial\phi}{\partial t} + \frac{\partial^{2.5}\phi}{\partial p^{2.5}} = \frac{\exp(2.55t-p)\left(2\exp(2.5t)+|\phi|+\left|\frac{\partial\phi}{\partial p}\right|\right)}{\left((p-0.05t)^2+3\ln(p-0.05t+e^2)\right)\left(\exp(2.5t)+|\phi|+\left|\frac{\partial\phi}{\partial p}\right|\right)}, & (p,t)\in\mathcal{H},\\ \phi(0.05t,t) = \frac{\partial\phi}{\partial p}(0.05t,t) = \frac{\partial^2\phi}{\partial p^2}(0.05t,t) = 0.\end{cases} \tag{14}$$

The transformation

$$\phi(p,t) = \exp(2.5t)y(z), \text{ with } z = p - 0.05t,$$

reduces IVP of GNFKdVEs (14) to this FDE

$$^{C}\mathcal{D}_{0^+}^{2.5}y(z) = -2.5y(z) + 0.05y'(z) + f\left(z, y(z), y'(z)\right), z \in [0,1],$$

with the conditions

$$y(0) = y'(0) = y''(0) = 0,$$

where

$$f\left(z, y(z), y'(z)\right) = \frac{\exp(-z)\left(2 + |y(z)| + |y'(z)|\right)}{\left(z^2 + 3\ln\left(z+e^2\right)\right)(1 + |y(z)| + |y'(z)|)}.$$

$\exp(-z)$, z^2 and $\ln(z+e^2)$ are continuous functions for any $z \in [0,1]$, then f is continuous. For any $y, x, \bar{y}, \bar{x} \in \mathbb{C}$ and $z \in [0,1]$, we get

$$\left|f(z,y,x) - f(z,\bar{y},\bar{x})\right| \le \frac{1}{6}|y-\bar{y}| + \frac{1}{6}|x-\bar{x}|.$$

Therefore, assumption ($\aleph$2) is satisfied with $\alpha = \beta = \frac{1}{6}$. In addition,

$$|f(z,y,x)| \le \frac{\exp(-z)}{z^2 + 3\ln\left(z+e^2\right)}(2 + |y| + |x|).$$

Thus, assumption ($\aleph$3) is satisfied with

$$\begin{aligned}u(z) &= \frac{2\exp(-z)}{z^2+3\ln\left(z+e^2\right)},\\ v(z) &= \frac{\exp(-z)}{z^2+3\ln\left(z+e^2\right)} \text{ and}\\ w(z) &= \frac{\exp(-z)}{z^2+3\ln\left(z+e^2\right)}.\end{aligned}$$

We also have

$$u^* = \frac{1}{3},\ v^* = \frac{1}{6} \text{ and } w^* = \frac{1}{6},$$

for

$$\theta = \max\{\beta, w^*\} = \frac{1}{6}.$$

Condition (11) gives

$$\Gamma(\mu) \simeq 1.3293 > \frac{\mu(\delta+\theta+\lambda)+\upsilon^*\lambda}{\mu\lambda^{1-\mu}} \simeq 1.2833.$$

It follows from Theorem 8 that IVP of GNFKdVEs (14) has at least one solution.

Theorem 9. *Assume that assumptions* ($\aleph 1$) *and* ($\aleph 2$) *hold. If we put* $\lambda < \left(\frac{\Gamma(\mu)}{\delta+\beta}\right)^{\frac{1}{\mu-1}}$ *and*

$$\frac{(\mu+\alpha)\lambda^\mu}{\Gamma(\mu+1)-\mu(\delta+\beta)\lambda^{\mu-1}} < 1 \tag{15}$$

then IVP of GNFKdVEs (1) *has a unique solution in the traveling wave form* (2) *on* $\mathcal{H}$.

Proof. Similarly to the steps taken in the proof of Theorem 8, IVP of GNFKdVEs (1) is reduced to fractional-order's problem (6)–(7), which can be formulated as a fixed point problem (12).

Let $\varphi_1, \varphi_2 \in C(\mathbb{E}, \mathbb{C})$, then we get

$$\mathcal{B}\varphi_1(z) - \mathcal{B}\varphi_2(z) = \frac{1}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}\big(\psi_1(\tau)-\psi_2(\tau)\big)d\tau,$$

where

$$\begin{aligned}\psi_i(z) &= g\big(z,\varphi_i(z),\psi_i(z)\big) - \mu\varphi_i(z), g\big(z,\varphi_i(z),\psi_i(z)\big)\\ &= \delta\mathcal{J}_{0^+}^{\mu-1}\psi_i(z) + f\left(z,\varphi_i(z),\mathcal{J}_{0^+}^{\mu-1}\psi_i(z)\right),\ \text{for } i=1,2.\end{aligned}$$

Also,

$$\big|\mathcal{B}\varphi_1(z)-\mathcal{B}\varphi_2(z)\big| \le \frac{1}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}\big|\psi_1(\tau)-\psi_2(\tau)\big|d\tau \tag{16}$$

We have

$$\|\psi_1-\psi_2\|_\infty \le \frac{(\mu+\alpha)\Gamma(\mu)}{\Gamma(\mu)-(\delta+\beta)\lambda^{\mu-1}}\|\varphi_1-\varphi_2\|_\infty.$$

From (16), we find

$$\|\mathcal{B}\varphi_1-\mathcal{B}\varphi_2\|_\infty \le \frac{(\mu+\alpha)\lambda^\mu}{\Gamma(\mu+1)-\mu(\delta+\beta)\lambda^{\mu-1}}\|\varphi_1-\varphi_2\|_\infty.$$

Thus, according to (15), $\mathcal{B}$ is a contraction operator.

By applying Banach's contraction principle (see [25]), we infer that $\mathcal{B}$ has a unique fixed point, which serves as the unique solution to problem (6)–(7) on $\mathbb{E}$. Similarly, the existence and uniqueness of a traveling wave for IVP of GNFKdVEs (1) is established, provided that condition (15) holds true for any $(p,t)\in\mathcal{H}$. □

Illustrative example 2. We consider the IVP of GNFKdVEs

$$\begin{cases}\frac{\partial\phi}{\partial t} + \frac{\partial^{2.7}\phi}{\partial p^{2.7}} = \frac{\cos(p-3t)\left(2\exp(2.7t)+|\phi|+\left|\frac{\partial\phi}{\partial p}\right|\right)}{\exp(p-5.7t)\left(\exp(2.7t)+|\phi|+\left|\frac{\partial\phi}{\partial p}\right|\right)}, & (p,t)\in\mathcal{H},\\ \phi(3t,t)=2,\ \frac{\partial\phi}{\partial p}(3t,t) = \frac{\partial^2\phi}{\partial p^2}(3t,t)=0.\end{cases} \tag{17}$$

The transformation

$$\phi(p,t) = \exp(2.7t)y(z),\ \text{with } z = p-3t,$$

reduces IVP of GNFKdVEs (14) to this FDE

$${}^C\mathcal{D}_{0^+}^{2.5}y(z) = -2.7y(z)+3y'(z)+f\big(z,y(z),y'(z)\big), z\in\left[0,\frac{1}{2}\right],$$

with the conditions

$$y(0)=2,\ \text{and}\ y'(0)=y''(0)=0,$$

where

$$f\big(z,y(z),y'(z)\big) = \frac{\cos(z)\big(2+|y(z)|+|y'(z)|\big)}{\exp(z)(1+|y(z)|+|y'(z)|)}.$$

As $\exp(z)$ and $\cos(z)$ are continuous nonnegative functions for any $z\in\left[0,\frac{1}{2}\right]$, then f is continuous. For any $y,x,\overline{y},\overline{x}\in\mathbb{C}$ and $z\in\left[0,\frac{1}{2}\right]$, we get

$$\left|f(z,y,x)-f\big(z,\overline{y},\overline{x}\big)\right| \le |y-\overline{y}|+|x-\overline{x}|.$$

Therefore, assumption ($\aleph 2$) is satisfied with $\alpha=\beta=1$. We have

$$\lambda = \frac{1}{2} < \left(\frac{\Gamma(\mu)}{\delta+\beta}\right)^{\frac{1}{\mu-1}} \simeq 0.57139,$$

and the condition in (15)

$$\frac{(\mu+\alpha)\lambda^\mu}{\Gamma(\mu+1)-\mu(\delta+\beta)\lambda^{\mu-1}} \simeq 0.67261 < 1.$$

It follows from Theorem 9 that IVP of GNFKdVEs (17) has a unique solution.

4 | Ulam–Hyers Stability Results

In this section, we use Definitions 4 and 5 to study the stability of the equation in (1) on $\mathcal{H}$.

Let $\psi_h\in C(\mathbb{E},\mathbb{C})$ as a continuous function satisfying the functional equation

$$\psi_h(z) = g\big(z,h(z),\psi_h(z)\big)-\mu h(z),\ \text{with } h\in C(\mathbb{E},\mathbb{C}),$$

where $g:\mathbb{E}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ satisfies for any $h\in C(\mathbb{E},\mathbb{C})$ and $2<\mu\le 3$;

$$g\big(z,h(z),\psi_h(z)\big) = \delta\mathcal{J}_{0^+}^{\mu-1}\psi_h(z)+f\left(z,h(z),\mathcal{J}_{0^+}^{\mu-1}\psi_h(z)\right).$$

We also define

$$\phi_h(p,t) = \exp(\mu t)h(z)\ \text{, with } z = p-\delta t \tag{18}$$

Before proceeding, we present the following remark introduced in [26], followed by a lemma aimed to simplify subsequent calculations.

Remark 10. If $x\in C(\mathbb{E},\mathbb{C})$ is a solution of the inequality

$$\left|{}^C\mathcal{D}_{0^+}^{\mu}x(z) - \psi_x(z)\right| \le \varepsilon, \forall z\in\mathbb{E} \tag{19}$$

for some $\varepsilon>0$, then there exists $\ell\in C(\mathbb{E},\mathbb{C})$, such that

1. ${}^{C}\mathcal{D}^{\mu}_{0^+}x(z) = \psi_x(z) + \ell(z)$, for any $z \in \mathbb{E}$,
2. $|\ell(z)| \le \varepsilon$, for all $z \in \mathbb{E}$.

Lemma 11. *If $x \in C(\mathbb{E}, \mathbb{C})$ is the solution of the inequality (19), then there exists $\varepsilon > 0$ such that x will be the solution of the inequality:*

$$\left|x(z) - x(0) - zx'(0) - \frac{1}{2}z^2x''(0) - \mathcal{J}^{\mu}_{0^+}\psi_x(z)\right| \le \frac{\varepsilon\lambda^{\mu}}{\Gamma(\mu+1)}, \forall z \in \mathbb{E}.$$

Proof. If $x \in C(\mathbb{E}, \mathbb{C})$ is a solution of (19). Then from Remark 10, we have

$$\begin{cases} {}^{C}\mathcal{D}^{\mu}_{0^+}x(z) = \psi_x(z) + \ell(z), & z \in \mathbb{E}, \\ |\ell(z)| \le \varepsilon, & \varepsilon > 0, \end{cases}$$

hence

$$x(z) = x(0) + zx'(0) + \frac{1}{2}z^2x''(0) + \mathcal{J}^{\mu}_{0^+}\left(\psi_x + \ell\right)(z), \forall z \in \mathbb{E}.$$

Then, for all $z \in \mathbb{E}$, we get

$$\begin{aligned} &\left|x(z) - x(0) - zx'(0) - \frac{1}{2}z^2x''(0) - \mathcal{J}^{\mu}_{0^+}\psi_x(z)\right| \\ &= \left|\mathcal{J}^{\mu}_{0^+}\left(\psi_x + \ell\right)(z) - \mathcal{J}^{\mu}_{0^+}\psi_x(z)\right| \\ &\le \frac{1}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}\ell(\tau)d\tau \\ &\le \frac{\varepsilon\lambda^{\mu}}{\Gamma(\mu+1)}. \end{aligned}$$

Hence, the lemma is proved. □

Theorem 12. *If assumptions (ℵ1) and (ℵ2) hold, then equation (3) is Ulam–Hyers stable and consequently generalized Ulam–Hyers stable.*

Proof. Let ϕ_x be a traveling wave solution of inequality (4), that is, ϕ_x is $C^3(\mathcal{H}, \mathbb{C})$ in space p and $C^1(\mathcal{H}, \mathbb{C})$ in time t and satisfies

$$\left|\frac{\partial\phi_x}{\partial t} + \frac{\partial^{\mu}\phi_x}{\partial p^{\mu}} - \mathcal{F}\left(p, t, \phi_x, \frac{\partial\phi_x}{\partial p}\right)\right| \le \varepsilon, \text{ for each } \varepsilon > 0.$$

Consequently, after using the transformation (18) and Theorem 6, we can get easily

$$\left|{}^{C}\mathcal{D}^{\mu}_{0^+}x(z) - \psi_x(z)\right| \le \varepsilon, x \in C(\mathbb{E}, \mathbb{C}) \tag{20}$$

Let us denote by ϕ_y the unique traveling wave solution of the following IVP of GNFKdVEs

$$\begin{cases} \frac{\partial\phi_y}{\partial t} + \frac{\partial^{\mu}\phi_y}{\partial p^{\mu}} = \mathcal{F}\left(p, t, \phi_y, \frac{\partial\phi_y}{\partial p}\right), & (p,t) \in \mathcal{H}, \\ \phi_y(\delta t, t) = \exp(\mu t)x(0), & x(0) \in \mathbb{C},\ \delta \in \mathbb{R}_+, \\ \frac{\partial\phi_y}{\partial p}(\delta t, t) = \exp(\delta t)x'(0),\ \frac{\partial^2\phi_y}{\partial p^2}(\delta t, t) = \exp(\delta t)x''(0) & x'(0), x''(0) \in \mathbb{C}. \end{cases}$$

After using the transformation (18) and Theorem 9, we observe that $y \in C(\mathbb{E}, \mathbb{C})$ is also the unique solution of the problem

$${}^{C}\mathcal{D}^{\mu}_{0^+}y(z) = \psi_y(z), \text{ with } y^{(k)}(0) = x^{(k)}(0), \text{ for } k = 0, 1, 2 \tag{21}$$

Lemma 3 helps us to infer, after applying $\mathcal{J}^{\mu}_{0^+}$ to both sides of Equation (21), that

$$y(z) = x(0) + zx'(0) + \frac{1}{2}z^2x''(0) + \mathcal{J}^{\mu}_{0^+}\psi_y(z), \forall z \in \mathbb{E} \tag{22}$$

On the other hand, using Lemma 11 and (22) makes us obtain

$$\begin{aligned} |x(z) - y(z)| &= \left|x(z) - \left(x(0) + zx'(0) + \frac{1}{2}z^2x''(0) + \mathcal{J}^{\mu}_{0^+}\psi_y(z)\right)\right| \\ &= \left|x(z) - x(0) - zx'(0) - \frac{1}{2}z^2x''(0)\right. \\ &\quad \left. - \mathcal{J}^{\mu}_{0^+}\psi_x(z) + \mathcal{J}^{\mu}_{0^+}\left(\psi_x - \psi_y\right)(z)\right| \\ &\le \left|x(z) - x(0) - zx'(0) - \frac{1}{2}z^2x''(0) - \mathcal{J}^{\mu}_{0^+}\psi_x(z)\right| \\ &\quad + \left|\mathcal{J}^{\mu}_{0^+}\left(\psi_x - \psi_y\right)(z)\right| \\ &\le \frac{\varepsilon\lambda^{\mu}}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}\left|\psi_x(\tau) - \psi_y(\tau)\right|d\tau. \end{aligned} \tag{23}$$

By (ℵ2), we have for each $z \in \mathbb{E}$

$$\begin{aligned} &\left|\psi_x(z) - \psi_y(z)\right| \\ &\le \mu|x(z) - y(z)| + \left|g\left(z, x(z), \psi_x(z)\right) - g\left(z, y(z), \psi_y(z)\right)\right|, \end{aligned}$$

with

$$\begin{aligned} &\left|g\left(z, x(z), \psi_x(z)\right) - g\left(z, y(z), \psi_y(z)\right)\right| \\ &\le \left|\delta\mathcal{J}^{\mu-1}_{0^+}\left(\psi_x - \psi_y\right)(z)\right| \\ &\quad + \left|f\left(z, x(z), \mathcal{J}^{\mu-1}_{0^+}\psi_x\right) - f\left(z, y(z), \mathcal{J}^{\mu-1}_{0^+}\psi_y\right)\right| \\ &\le (\delta+\beta)\left|\mathcal{J}^{\mu-1}_{0^+}\left(\psi_x - \psi_y\right)(z)\right| + \alpha|x(z) - y(z)|. \end{aligned}$$

Then

$$\begin{aligned} \left|\psi_x(z) - \psi_y(z)\right| &\le (\mu+\alpha)|x(z) - y(z)| \\ &\quad + \frac{\delta+\beta}{\Gamma(\mu-1)}\int_0^z (z-\tau)^{\mu-2}\left|\psi_x(\tau) - \psi_y(\tau)\right|d\tau. \end{aligned}$$

By using Gronwall's inequality [27], we get

$$\left|\psi_x(z) - \psi_y(z)\right| \le \kappa|x(z) - y(z)|, \forall z \in \mathbb{E}, \tag{24}$$

where $\kappa = (\mu+\alpha)\exp\left(\frac{\delta+\beta}{\lambda^{1-\mu}\Gamma(\mu)}\right)$. Thus, by replacing (24) in (23), we get

$$|x(z)-y(z)| \leq \frac{\varepsilon\lambda^{\mu}}{\Gamma(\mu+1)} + \frac{\kappa}{\Gamma(\mu)}\int_0^z (z-\tau)^{\mu-1}|x(\tau)-y(\tau)|d\tau,$$

and using Gronwall's inequality gives us

$$|x(z)-y(z)| \leq \frac{\lambda^{\mu}\varepsilon}{\Gamma(\mu+1)}\exp\left(\frac{\kappa\lambda^{\mu}}{\Gamma(\mu+1)}\right),$$

In another way,

$$\left|\phi_x(p,t)-\phi_y(p,t)\right| = |\exp(\mu t)(x(z)-y(z))| \leq \gamma\varepsilon,$$

where $\gamma = \frac{\lambda^{\mu}\varepsilon}{\Gamma(\mu+1)}\exp\left(\frac{\kappa\lambda^{\mu}}{\Gamma(\mu+1)}+\mu T\right)$. Definition 4 helps us infer that equation (3) is Ulam–Hyers stable on $\mathcal{H}$. This completes the proof.

If we select $Q(\varepsilon)=\gamma\varepsilon$, it follows that $Q(0)=0$. Subsequently, according to Definition 5, it can be inferred that equation (3) manifests generalized Ulam–Hyers' stability. □

Remark 11. We can observe that both equations in problems (14) and (17) manifest Ulam–Hyers' stability.

5 | Illustration With Numerical Simulation

The KdV equation describes various wave phenomena across scientific fields. Its solutions include periodic waveforms, modeled by trigonometric functions, and solitons [4, 8, 9], which maintain their shape and speed over long distances. Periodic solutions model surface water waves, internal ocean waves, and plasma wave propagation. Solitons are crucial for understanding solitary waves in shallow water, optical pulses in fiber optics, and wave dynamics in elastic media. The equation's ability to capture the interplay between nonlinearity and dispersion makes it essential for predicting and analyzing wave behavior in fluid dynamics, coastal engineering, and plasma physics.

Here, we provide an explicit solution in the form of a traveling wave, modeled by trigonometric functions for IVP of GNFKdVEs (1).

Let $2<\mu\leq 3$ and $\omega\in\mathbb{C}$ be a constant that depends on μ, which we refer to as the"wavelength." Then

$$y(z)=\frac{2}{\mu^2}(\exp(\omega z)+\cos(\omega z)-\sin(\omega z)),$$

is a solution of problem (6)–(7), where

$$\begin{aligned} &f\left(z,y(z),y'(z)\right)\\ &=\frac{2\omega^3 z^{3-\mu}}{\mu^2}\left[E_{1,4-\mu}(\omega z)-\frac{i+1}{2}E_{1,4-\mu}(i\omega z)\right.\\ &\left.+\frac{i-1}{2}E_{1,4-\mu}(-i\omega z)\right]+\frac{2}{\mu}y(z)-\frac{2\omega\delta}{\mu^2}y'(z). \end{aligned}$$

Then the traveling wave solution for IVP of GNFKdVEs (1) is given by

$$\begin{aligned} \phi(p,t)=\frac{2}{\mu^2}\exp(\mu t)(\exp(\omega(p-\delta t))\\ +\cos(\omega(p-\delta t))-\sin(\omega(p-\delta t))), \end{aligned} \tag{25}$$

where

$$\begin{aligned} &\mathcal{F}\left(p,t,\phi,\frac{\partial\phi}{\partial p}\right)\\ &=\frac{2\omega^3(p-\delta t)^{3-\mu}}{\mu^2\exp(-\mu t)}\left[E_{1,4-\mu}(\omega(p-\delta t))-\frac{i+1}{2}E_{1,4-\mu}(i\omega(p-\delta t))\right.\\ &\left.+\frac{i-1}{2}E_{1,4-\mu}(-i\omega(p-\delta t))\right]+\mu\phi(p,t)-\delta\frac{\partial\phi}{\partial p}. \end{aligned}$$

The conditions on μ, δ, and ω in the solution (25) describe the amplitude, energy, wavelength, frequency, and propagation characteristics of the waves. These parameters collectively capture the intricate balance between nonlinear and dispersive effects that shape the wave phenomena modeled by the KdV equation.

For $\delta=\frac{1}{1000}$ and μ taking different values in the interval $(2,3]$, we provide two figures that describe various wave characteristics. These characteristics, which include wave width, wave height, and the temporal depth of wave formation (TDWF), will be explained in detail under each figure.

1. When the wavelength ω decreases as μ increases, we put:

$$\omega(\mu)=\frac{2}{25}(1-2\mu).$$

   Solution (25) can be represented by the following figure:

   - Figure 1 illustrates the wave's spatial and temporal evolution. As μ increases, the wave height also increases due to the combined effects of a higher wave number (shorter wavelength) and nonlinear amplification. This results in more waves fitting into a given spatial interval, leading to an increase in the number of waves. Additionally, the amplitude increases with the wave number, which can be attributed to higher energy levels associated with shorter wavelengths in nonlinear wave equations like the KdV equation.

   - GNFKdVE (1) includes a nonlinear term $\mathcal{F}$, which amplifies the wave amplitude as the wave number increases. This phenomenon, known as wave steepening, results from more pronounced nonlinear interactions at higher μ values. The dispersion relation δ describes how the phase speed depends on the wave number, with higher wave numbers corresponding to faster phase speeds. Consequently, the decreasing wavelength leads to a shorter wave period, increasing the wave's steepness and amplitude.

   These points collectively explain why increasing the fractional order μ results in more waves with higher amplitudes in the context of your solution to the equation.

2. When the wavelength ω increases as μ increases, for example:

$$\omega(\mu)=\frac{1}{\pi}\left(\mu-\frac{4\pi^2}{11}\right).$$

   Solution (25) is represented as follows:

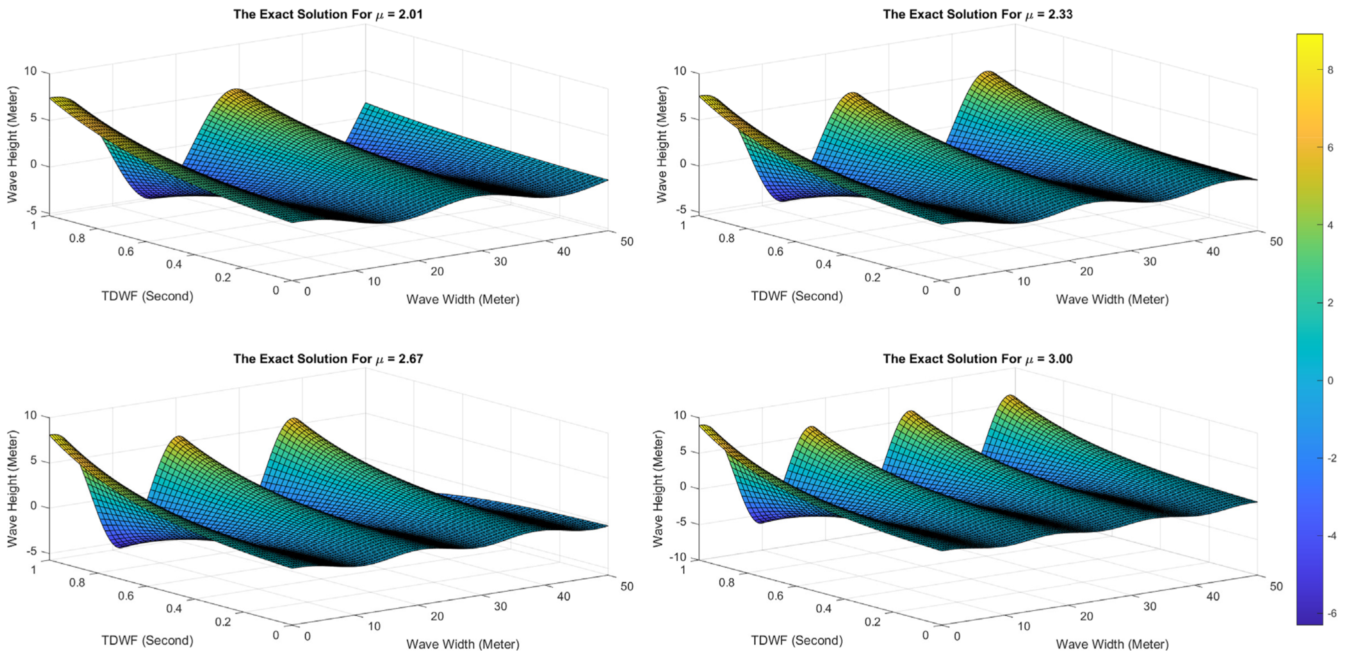

FIGURE 1 | Effect of decreasing the wavelength ω simultaneously with increasing the fractional order μ on the characteristics of traveling wave solution (25). [Colour figure can be viewed at wileyonlinelibrary.com]

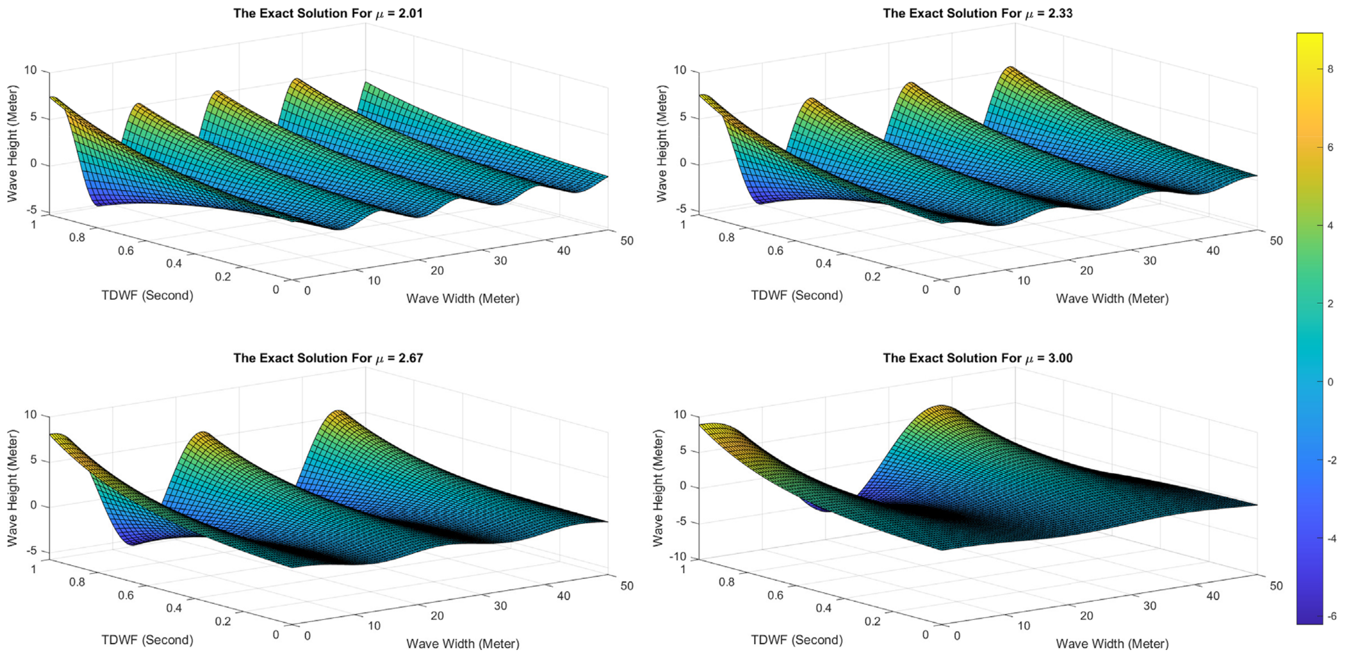

FIGURE 2 | Effect of increasing the wavelength ω simultaneously with increasing the fractional order μ on the characteristics of traveling wave solution (25). [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 2 illustrates that as the fractional order μ increases, the wavelength ω also increases, leading to a decrease in the wave number. This results in fewer waves within a given spatial interval and lower amplitudes. Although the energy of the waves may increase slightly, the nonlinear effects become less significant. This behavior is depicted in the graphical representation, which shows fewer but larger waves as the wavelength increases.

6 | Conclusion

This paper has explored the existence and uniqueness of traveling wave solutions for a generalized nonlinear fractional Korteweg–de Vries equation using several fixed-point theorems. Stability was further examined through the Ulam–Hyers stability theorem, demonstrating the robustness of the results under small perturbations.

Through detailed examples and explicit solutions, we highlighted the interplay between nonlinearity and dispersion in wave phenomena, showing how parameters like amplitude and wave number influence propagation characteristics. The traveling wave method proved to be an effective analytical tool, simplifying the fractional partial differential equation and providing deeper insights into its dynamics.

These findings enhance the understanding of fractional-order Korteweg–de Vries equations and illustrate the utility of fractional calculus in modeling complex phenomena with memory effects. However, the assumptions underlying our model require further exploration to ensure its generalizability. Future work will focus on analyzing these assumptions, comparing them with those in related studies, and assessing their implications for broader applications, such as systems with different boundary conditions or additional complexities. This effort will not only address potential limitations but also pave the way for refining the model to better capture real-world scenarios and further enrich the field.

Author Contributions

Rabah Djemiat: conceptualization, investigation, funding acquisition, writing – original draft, software, formal analysis. **Sami Galleze:** conceptualization, investigation, funding acquisition, writing – original draft, software, formal analysis. **Bilal Basti:** methodology, validation, visualization, writing – review and editing, project administration, supervision, resources, data curation.

References

1. I. Podlubny, *Fractional Differential Equations*. Mathematics in Science and Engineering (Academic Press, 1999).

2. A. A. Kilbas, H. H. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier Science B.V, 2006).

3. K. Diethelm, *The Analysis of Fractional Differential Equations* (Springer, 2010).

4. A. R. Seadawy, K. A. Khalid, and R. I. Nuruddeen, “A Variety of Soliton Solutions for the Fractional Wazwaz-Benjamin-Bona-Mahony Equations,” *Results in Physics* 12 (2019): 2234–2241.

5. A. R. Seadawy, M. Arshad, and D. Lu, “The Weakly Nonlinear Wave Propagation Theory for the Kelvin-Helmholtz Instability in Magnetohydrodynamics Flows,” *Chaos, Solitons & Fractals* 139 (2020): 110141.

6. A. R. Seadawy, “Stability Analysis for Zakharov-Kuznetsov Equation of Weakly Nonlinear Ion-Acoustic Waves in a Plasma,” *Computers & Mathematcs with Applications* 67, no. 1 (2014): 172–180.

7. X. Yin, L. Xu, and L. Yang, “Evolution and Interaction of Soliton Solutions of Rossby Waves in Geophysical Fluid Mechanics,” *Nonlinear Dynamics* 111, no. 13 (2023): 12433–12445.

8. N. Cao, X. Yin, and S. Bai, “Breather Wave, Lump Type and Interaction Solutions for a High Dimensional Evolution Model,” *Chaos, Solitons & Fractals* 172 (2023): 113505.

9. L. Xu, X. Yin, N. Cao, and S. Bai, “Multi-Soliton Solutions of a Variable Coefficient Schrödinger Equation Derived From Vorticity Equation,” *Nonlinear Dynamics* 112, no. 3 (2024): 2197–2208.

10. Z. Xing, L. Wen, and W. Wang, “An Efficient Difference Scheme for Time-Fractional KdV Equation,” *Computational and Applied Mathematics* 40 (2021): 277.

11. D. J. Korteweg and G. de Vries, “On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves,” *Philosophical Magazine* 39, no. 5 (1895): 422–443.

12. O. Ersoy and I. Dag, “The Exponential Cubic B-Spline Algorithm for Korteweg-de Vries Equation,” *Advances in Numerical Analysis* 2015 (2015): 367056.

13. M. M. A. Khater, “Advanced Computational Techniques for Solving the Modified KdV-KP Equation and Modeling Nonlinear Waves,” *Optical and Quantum Electronics* 56 (2024): 6.

14. B. Basti, R. Djemiat, and N. Benhamidouche, “Theoretical Studies on the Existence and Uniqueness of Solutions for a Multidimensional Nonlinear Time and Space-Fractional Reaction-Diffusion/Wave Equation,” *Memoirs on Differential Equations and Mathematical Physics* 89 (2023): 1–16.

15. B. Basti and N. Benhamidouche, “Existence Results of Self-Similar Solutions to the Caputo-type’s Space-Fractional Heat Equation,” *Surveys in Mathematics and Its Applications* 15 (2020): 153–168.

16. B. Basti and N. Benhamidouche, “Global Existence and Blow-Up of Generalized Self-Similar Solutions to Nonlinear Degenerate Diffusion Equation Not in Divergence Form,” *Applied Mathematics E-Notes* 20 (2020): 367–387.

17. N. Benhamidouche, “Exact Solutions to Some Nonlinear PDEs, Travelling Profiles Method,” *Electronic Journal of Qualitative Theory of Differential Equations* 15 (2008): 1–7.

18. E. Buckwar and Y. Luchko, “Invariance of a Partial Differential Equation of Fractional Order Under the lie Group of Scaling Transformations,” *Journal of Mathematical Analysis and Applications* 227, no. 1 (1998): 81–97.

19. L. Debnath and D. D. Bhatta, “Solutions to Few Linear Fractional Inhomogeneous Partial Differential Equations in Fluid Mechanics,” *Fractional Calculus and Applied Analysis* 7, no. 1 (2004): 21–36.

20. R. Djemiat, B. Basti, and N. Benhamidouche, “Analytical Studies on the Global Existence and Blow-Up of Solutions for a Free Boundary Problem of Two-Dimensional Diffusion Equations of Moving Fractional Order,” *Advances in the Theory of Nonlinear Analysis and Its Application* 6, no. 8 (2022): 287–299.

21. R. Djemiat, B. Basti, and N. Benhamidouche, “Existence of Traveling Wave Solutions for a Free Boundary Problem of a Higher-Order Space-Fractional Wave Equation,” *Applied Mathematics E-Notes* 22 (2022): 427–436.

22. R. Djemiat, B. Basti, and N. Benhamidouche, “Cauchy Problem for Jordan-Moore-Gibson-Thompson Equations of Nonlinear Acoustics With Fractional Operators,” *Analele Stiintifice ale Universitatii Al I Cuza din Iasi Matematica* 69, no. 2 (2023): 143–161.

23. Y. Luchko and R. Gorenfl, “Scale-Invariant Solutions of a Partial Differential Equation of Fractional Order,” *Fractional Calculus and Applied Analysis* 1, no. 1 (1998): 63–78.

24. F. Nouioua and B. Basti, “Global Existence and Blow-Up of Generalized Self-Similar Solutions for a Space-Fractional Diffusion Equation With Mixed Conditions,” *Ann. Univ. Paedagog. Crac. Stud. Math.* 20 (2020): 43–56.

25. A. Granas and J. Dugundji, *Fixed Point Theory* (Springer-Verlag, 2003).

26. A. L. Ï. Zeeshan, Z. A. D. A. Akbar, and S. H. A. H. Kamal, “Ulam Stability Results for the Solutions of Nonlinear Implicit Fractional Order Differential Equations,” *Hacettepe Journal of Mathematics and Statistics* 48, no. 4 (2018): 1092–1109.

27. C. Corduneanu, *Principles of Differential and Integral Equations* (Allyn and Bacon, 1971).