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Compatibility and Traces of Bipolar Fuzzy Relations: Characterizations and Extensions

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Abstract

In this paper, we introduce Bělohl'avek's concept of compatibility within the context of bipolar fuzzy relations. We provide a comprehensive characterization of both left and right compatibility for these relations. Additionally, we explore several properties of traces in relation to bipolar fuzzy relations and offer novel characterizations of compatibility in terms of left and right traces. Building upon the work of De Baets et al., who studied the compatibility of fuzzy tolerance relations (and fuzzy equivalence relations in particular) with a given (strict) order relation, we extend their analysis by examining the left- and right-compatibility of bipolar fuzzy tolerance (or equivalence) relations with bipolar characteristic functions of (strict) order relations.

Keywords: Left compatibility, right compatibility, traces, order relation, bipolar fuzzy relation, bipolar fuzzy tolerance relation.

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1.|Introduction

Since Zadeh introduced the concept of fuzzy sets in 1965 [35], extensive research has focused on extending classical set-theoretic notions into the fuzzy setting. Among these advancements, the concept of bipolar fuzzy sets, introduced by Zhang [37], has attracted significant attention for its ability to represent both positive and negative degrees of membership within a unified framework. Building on this foundation, K. M. Lee [26] investigated the relationships among interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets. Further contributions include Lee's application of bipolar fuzzy sets to the study of BCK/BCI-algebras [25]. More recently, Jeong-Gon Lee and Kul Hur [24] introduced the notions of bipolar fuzzy reflexive, symmetric, and transitive relations, thereby extending classical relation theory into the bipolar fuzzy context.

The notion of compatibility between a binary fuzzy relation and a fuzzy equality relation was first introduced by Bělohlávek [8]. Later, Kheniche et al. [22] demonstrated that this notion is equivalent to the earlier concept of extensionality for mappings between universes endowed with fuzzy equalities, originally introduced by Höhle and Blanchard [20]. Since then, compatibility has become a fundamental tool in the study of fuzzy functions [13, 30], fuzzy ordered structures [2, 3, 6, 8, 14, 38, 39], and fuzzy algebras [13]. For further developments and applications, see for example [1, 9, 15, 21, 28, 29, 31, 32, 33].

A notable extension of compatibility theory was proposed by De Baets et al. [12], who investigated the compatibility of strict order relations with fuzzy relations, and characterized fuzzy tolerance (or fuzzy equivalence) relations that preserve compatibility with a given strict order. Building on this foundation, Zedam et al. [36] introduced the notions of left- and right-compatibility, providing a detailed characterization of fuzzy tolerance relations compatible with strict orders. Furthermore, Omar Barkat et al. [5] examined key properties concerning the compatibility of ternary relations with binary fuzzy relations, thereby enriching the theoretical landscape of fuzzy relational structures.

Structure of the Paper. This paper is organized as follows:

- Section 2 provides a review of foundational concepts, including triangular norms, the residual implication associated with left-continuous t-norms, bipolar fuzzy relations, and the core properties of bipolar fuzzy compatibility.
- Section 3 extends Bělohlávek's notion of compatibility to the bipolar fuzzy framework. In this context, we introduce the concepts of left-compatibility and right-compatibility, and establish several key theoretical results.
- Section 4 explores the structural properties of the traces of bipolar fuzzy relations, and presents various characterizations of compatibility. In particular, we study the left- and right-compatibility of bipolar fuzzy tolerance relations with the characteristic function of an order relation, showing that these compatibility notions are generally trivial when applied to characteristic functions of arbitrary order relations.
- Section 5 focuses on the special case of characteristic functions of strict order relations, where additional structural insights and compatibility behaviors are investigated.

2.|Basic concepts

2.1.|Triangular Norms

A triangular norm (or t-norm) is a binary fuzzy operation T on the unit interval $[0,1]$ that is commutative, associative, monotone, and has 1 as a neutral element. A t-norm T is called left-continuous if and only if all partial mappings $T(x,\cdot)$ and $T(\cdot,x)$ are left-continuous.

If for two t-norms T_1 and T_2 we have $T_1(x,y) \leq T_2(x,y)$ for all $(x,y) \in [0,1]^2$, we say that T_2 is stronger than T_1, and we write $T_1 \leq T_2$ (see, e.g., [10, 11, 23]).

Example 1. *The following are the four basic t-norms:*

- *Minimum t-norm:* $T_M(x, y) = \min(x, y)$
- *Product t-norm:* $T_p(x, y) = xy$
- *Łukasiewicz t-norm:* $T_L(x, y) = \max(x + y - 1, 0)$
- *Drastic product t-norm:*

$$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[\times[0, 1[\\ x & \text{if } y = 1 \\ y & \text{if } x = 1 \end{cases}$$

Remark 2. *At the point* $(x, y) = (0.9, 1)$*, all four basic t-norms yield the same value:*

$$T_D = T_L = T_p = T_M = 0.9$$

Hence, the general inequality

$$T_D < T_L < T_p < T_M$$

does not hold pointwise at this specific input.

For a left-continuous t-norm T, the residual implication (residuum) $\overrightarrow{T}$ is defined as:

$$\overrightarrow{T}(x, y) = \sup \{u \in [0, 1] \mid T(u, x) \leq y\}$$

If T is a left-continuous t-norm, the following properties hold for all $x, y, z \in [0, 1]$ (see [17, 18, 19]):

(1) $x \leq y$ if and only if $\overrightarrow{T}(x, y) = 1$

(2) $T(x, y) \leq z$ if and only if $x \leq \overrightarrow{T}(y, z)$

(3) $T(\overrightarrow{T}(x, y), \overrightarrow{T}(y, z)) \leq \overrightarrow{T}(x, z)$

(4) $\overrightarrow{T}(1, y) = y$

(5) $T(x, \overrightarrow{T}(x, y)) \leq y$

(6) $y \leq \overrightarrow{T}(x, T(x, y))$

Throughout this paper, the notations $(\leq, \wedge, \vee)$ will refer to the usual order, minimum, and maximum operations on the real interval $[0, 1]$, respectively. Additionally, the notation $|.|$ will refer to the absolute value in the unit interval $[0, 1]$.

2.2.|Bipolar fuzzy set

Let X be a non-empty set. A pair $A = (A^-, A^+)$ is called a bipolar-valued fuzzy set (or bipolar fuzzy set) in X, if $A^- : X \longrightarrow [-1, 0]$ and $A^+ : X \longrightarrow [0, 1]$ are mappings. We denote by $BFS_S(X)$ the set of all bipolar fuzzy sets on X. Specifically, the bipolar fuzzy empty set and the bipolar fuzzy whole set in X are denoted by $\tilde{0}$ and $\tilde{1}$, respectively, where $\tilde{0}(x) = (0, 0)$ and $\tilde{1}(x) = (-1, 1)$ for each $x \in X$ [24].

Example 3. *Consider a fuzzy set "young" defined on the age domain* $[0, 100]$ *(see Fig. 1 in [26]). Now consider two ages 50, and 95, with membership degree 0. Although both of them do not satisfy the property "young", we may say that age 95 is more apart from the property rather than age 50 (see [26]).*

The negative membership degrees represent the degree to which elements satisfy an implicit counter-property (e.g., "old" against the property "young"). This bipolar-valued fuzzy set representation allows elements that have a membership degree of 0 in traditional fuzzy sets to be classified into two categories: those with a membership degree of 0 (irrelevant elements) and those with negative membership degrees (contrary elements). For example, the age values 50 and 95, which both have a membership degree of 0 in the fuzzy sets shown in Figure 1, are represented with membership degrees of 0 and negative values in the bipolar-valued fuzzy set of Figure 2, respectively. This

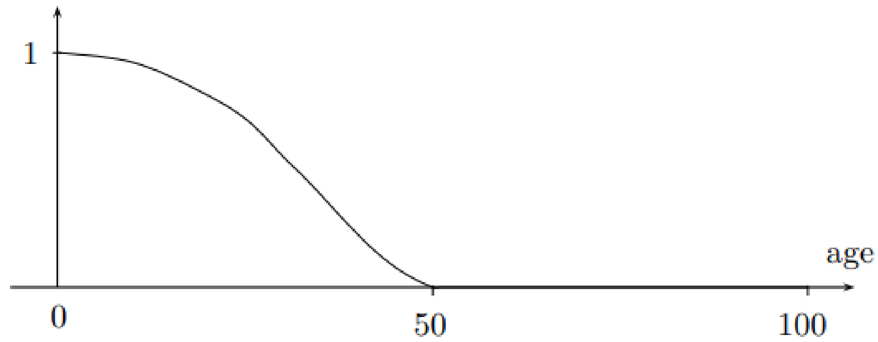

Figure 1. A fuzzy set "young".

distinction clearly indicates that age 50 is irrelevant to the property "young," while age 95 is further removed from the property "young," signifying that 95 is contrary to the property "young" (see [26]).

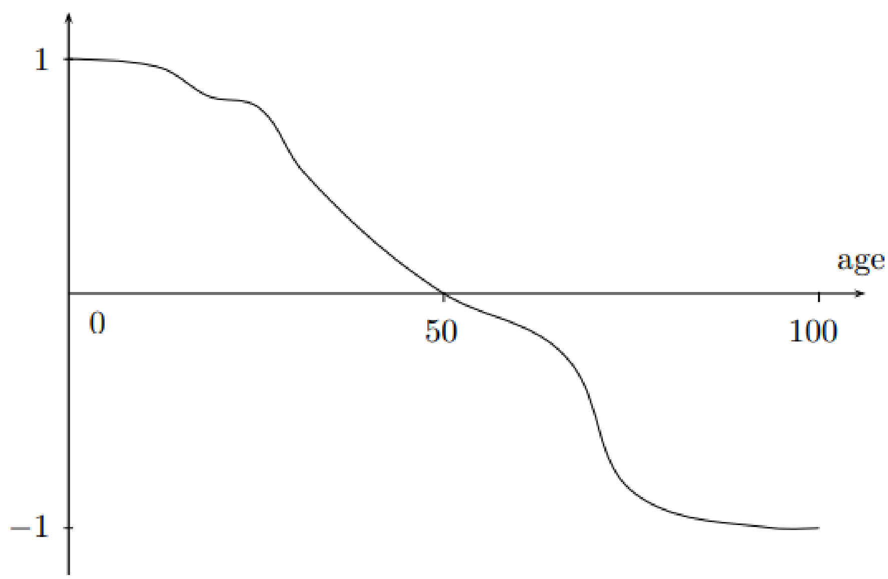

Figure 2. A bipolar fuzzy set "young".

It is evident that for each $A \in BFS_S(X)$ and $x \in X$, if $0 \leq A^+(x) - A^-(x) \leq 1$, then A is an intuitionistic fuzzy set as introduced by Atanassov [4]. In fact, $A^+(x)$ (resp. $-A^-(x)$) denotes the membership degree (resp. non-membership degree) of x in A. For any subset A of X, the bipolar fuzzy characteristic function of A is defined as the BFS $\chi_A = \langle x, \chi_A^-, \chi_A^+ \rangle$ in X, where:

$$\chi_A^-(x) = \begin{cases} -1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A, \end{cases} \quad \text{and} \quad \chi_A^+(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

The following expressions are defined for all bipolar fuzzy sets A, B in X [27]:

(1) $A \subseteq B$ if and only if $A^+ \leq B^+$ and $A^- \geq B^-$;

(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;

(3) $A^c = (-1 - A^-, 1 - A^+)$;

(4) $A \cap B = (A^- \vee B^-, A^+ \wedge B^+)$;

(5) $A \cup B = (A^- \wedge B^-, A^+ \vee B^+)$.

2.3.|Bipolar fuzzy relations

A bipolar fuzzy relation (BFR) ρ is a bipolar fuzzy subset of $X \times Y$ given by:

$$\rho = \{\langle (x, y), \rho^-(x, y), \rho^+(x, y)\rangle \mid x \in X, y \in Y\}$$

where $\rho^- : X \times Y \longrightarrow [-1, 0]$ and $\rho^+ : X \times Y \longrightarrow [0, 1]$ for every $(x, y) \in X \times Y$. If ρ is a bipolar fuzzy relation from X to itself, then ρ is called a bipolar fuzzy relation on X, and we denote the set of all bipolar fuzzy relations on X as $BFR(X)$ [24, 34].

The transpose ρ^t of a bipolar fuzzy relation ρ is defined as:

$$\rho^t = \left((\rho^-)^t, (\rho^+)^t\right)$$

where, for any $x, y \in X$, we have $(\rho^-)^t(x, y) = \rho^-(y, x)$ and $(\rho^+)^t(x, y) = \rho^+(y, x)$.

If $R = (R^-, R^+)$ is a bipolar fuzzy relation in a non-empty set X, then:

(1) R is called reflexive if $R^-(x, x) = -1$ and $R^+(x, x) = 1$, for all $x \in X$;

(2) R is called symmetric if $R^-(x, y) = R^-(y, x)$ and $R^+(x, y) = R^+(y, x)$, for all $x, y \in X$;

(3) R is called transitive if $(R^+(x, y) \wedge R^+(y, z) \leq R^+(x, z))$ and $(|R^-(x, y)| \wedge |R^-(y, z)| \leq |R^-(x, z)|)$, for all $x, y, z \in X$.

A bipolar fuzzy reflexive and transitive relation R on X is called a bipolar fuzzy preordering relation. A bipolar fuzzy symmetric preordering relation is called a bipolar fuzzy equivalence relation.

If R is a bipolar fuzzy relation on X, E is an equivalence relation on X, and T is a triangular norm, then:

(i) R is called E-reflexive if $R^-(x, y) \leq E^-(x, y)$ and $R^+(x, y) \geq E^+(x, y)$, for all $x, y \in X$;

(ii) R is called T-transitive if $T(|R^-(x, y)|, |R^-(y, z)|) \leq |R^-(x, z)|$ and $T(R^+(x, y), R^+(y, z)) \leq R^+(x, z)$, for all $x, y, z \in X$.

A bipolar fuzzy reflexive and T-transitive relation R on X is called a bipolar fuzzy T-preordering relation. A bipolar fuzzy symmetric T-preordering relation is called a bipolar fuzzy T-equivalence relation.

3.|Left and Right Compatibility of Bipolar Fuzzy Relations

Bělohlávek [7] introduced the concept of compatibility for L-relations with respect to an L-equality relation on a universe set X. In this section, we generalize this concept to bipolar fuzzy relations.

Definition 4. *Let ρ_1 and ρ_2 be two bipolar fuzzy relations on a universe set X, and let T be a t-norm.*

(i) *We say that ρ_1 is* left compatible *with ρ_2, denoted $\rho_1 \bigtriangledown_l^T \rho_2$, if for all $x, y, z \in X$, the following conditions hold:*

$$T\left(\left|\rho_1^-(x, y)\right|, \left|\rho_2^-(x, z)\right|\right) \leq \left|\rho_1^-(z, y)\right| \quad \text{and} \quad T\left(\rho_1^+(x, y), \rho_2^+(x, z)\right) \leq \rho_1^+(z, y)$$

(*ii*) *We say that* ρ_1 *is* right compatible *with* ρ_2*, denoted* $\rho_1 \bigtriangledown_r^T \rho_2$*, if for all* $x, y, t \in X$*, the following conditions hold:*

$$T\left(\left|\rho_1^-(x,y)\right|, \left|\rho_2^-(y,t)\right|\right) \leq \left|\rho_1^-(x,t)\right| \quad \textit{and} \quad T\left(\rho_1^+(x,y), \rho_2^+(y,t)\right) \leq \rho_1^+(x,t)$$

(*iii*) *We say that* ρ_1 *is* compatible *with* ρ_2*, denoted* $\rho_1 \bigtriangledown^T \rho_2$*, if for all* $x, y, z, t \in X$*, the following conditions hold:*

$$T\left(\left|\rho_1^-(x,y)\right|, T\left(\left|\rho_2^-(x,z)\right|, \left|\rho_2^-(y,t)\right|\right)\right) \leq \left|\rho_1^-(z,t)\right|$$
$$T\left(\rho_1^+(x,y), T\left(\rho_2^+(x,z), \rho_2^+(y,t)\right)\right) \leq \rho_1^+(z,t)$$

Lemma 5. *Let* ρ_1 *and* ρ_2 *be two bipolar fuzzy relations on a universe* X*, and let* T *be a t-norm. Then the following equivalences hold:*

(1) $\rho_1 \bigtriangledown_l^T \rho_2$ *if and only if* $\rho_1^t \bigtriangledown_r^T \rho_2$

(2) $\rho_1 \bigtriangledown_r^T \rho_2$ *if and only if* $\rho_1^t \bigtriangledown_l^T \rho_2$

(3) $\rho_1 \bigtriangledown^T \rho_2$ *if and only if* $\rho_1^t \bigtriangledown_l^T \rho_2$

Proof:

(1): Let $x, y, z \in X$. Assume that $\rho_1 \bigtriangledown_l^T \rho_2$ holds. By definition, we have:

$$T\left(\left|\rho_1^-(x,y)\right|, \left|\rho_2^-(x,z)\right|\right) \leq \left|\rho_1^-(z,y)\right| \quad \text{and} \quad T\left(\rho_1^+(x,y), \rho_2^+(x,z)\right) \leq \rho_1^+(z,y)$$

Taking the transpose of ρ_1, we use the identities:

$$(\rho_1^t)^+(y,x) = \rho_1^+(x,y) \quad \text{and} \quad (\rho_1^t)^-(y,x) = \rho_1^-(x,y)$$

Then, we obtain:

$$T\left(\left|(\rho_1^t)^-(y,x)\right|, \left|\rho_2^-(x,z)\right|\right) = T\left(\left|\rho_1^-(x,y)\right|, \left|\rho_2^-(x,z)\right|\right) \leq \left|\rho_1^-(z,y)\right| = \left|(\rho_1^t)^-(y,z)\right|$$

and similarly:

$$T\left((\rho_1^t)^+(y,x), \rho_2^+(x,z)\right) = T\left(\rho_1^+(x,y), \rho_2^+(x,z)\right) \leq \rho_1^+(z,y) = (\rho_1^t)^+(y,z)$$

Hence, $\rho_1^t \bigtriangledown_r^T \rho_2$ holds.

(2): The proof is analogous to (1) by reversing the roles of left and right compositions and applying the transpose accordingly.

(3): By definition, $\rho_1 \bigtriangledown^T \rho_2$ means that both $\rho_1 \bigtriangledown_l^T \rho_2$ and $\rho_1 \bigtriangledown_r^T \rho_2$ hold. Applying the equivalences from (1) and (2), we conclude:

$$\rho_1 \bigtriangledown^T \rho_2 \quad \Longleftrightarrow \quad \rho_1^t \bigtriangledown_r^T \rho_2 \quad \text{and} \quad \rho_1^t \bigtriangledown_l^T \rho_2 \quad \Longleftrightarrow \quad \rho_1^t \bigtriangledown_l^T \rho_2$$

as required. □

Remark 6. *In Lemma 5, it may seem that taking the transpose of* ρ_1 *causes a reversal or preservation of the inequality. However, the equivalence actually stems from the symmetry of the composition definitions under transposition: the left composition of* ρ_1 *becomes the right composition of* ρ_1^t*, since for all* $x, y \in X$*, we have* $(\rho_1^t)^+(y,x) = \rho_1^+(x,y)$*, and similarly for the negative part. Thus, the t-norm inequality is structurally preserved. A numerical example confirms this equivalence.*

Example 7. *Let* $X = \{a, b, c\}$ *be a finite universe, and define two bipolar fuzzy relations* ρ_1 *and* ρ_2 *on* X *via their positive parts as follows:*

$$\rho_1^+ = \begin{bmatrix} 1 & 0.6 & 0.2 \\ 0.7 & 1 & 0.3 \\ 0.5 & 0.4 & 1 \end{bmatrix}, \qquad \rho_2^+ = \begin{bmatrix} 1 & 0.8 & 0.5 \\ 0.6 & 1 & 0.4 \\ 0.3 & 0.2 & 1 \end{bmatrix}$$

We use the product t-norm $T(x,y) = x \cdot y$*.*

Let us check whether the inequality

$$T\left(\rho_1^+(x,y), \rho_2^+(x,z)\right) \le \rho_1^+(z,y)$$

holds for the triple $(x,y,z) = (a,b,c)$.

We compute:

$$\rho_1^+(a,b) = 0.6, \quad \rho_2^+(a,c) = 0.5, \quad \rho_1^+(c,b) = 0.4$$
$$T(0.6, 0.5) = 0.3 \le 0.4 = \rho_1^+(c,b)$$

Hence, $\rho_1 \bigtriangledown_l^T \rho_2$ *holds at this point.*

Now consider the transpose ρ_1^t*, whose positive part satisfies:*

$$(\rho_1^t)^+(y,x) = \rho_1^+(x,y)$$

We test the condition for $\rho_1^t \bigtriangledown_r^T \rho_2$ *with the same triple:*

$$T\left((\rho_1^t)^+(b,a), \rho_2^+(a,c)\right) = T(0.6, 0.5) = 0.3 \le (\rho_1^t)^+(b,c) = \rho_1^+(c,b) = 0.4$$

Thus, the inequality also holds for $\rho_1^t \bigtriangledown_r^T \rho_2$*, illustrating that:*

$$\rho_1 \bigtriangledown_l^T \rho_2 \quad \Longleftrightarrow \quad \rho_1^t \bigtriangledown_r^T \rho_2$$

Proposition 8. *Let* ρ_1 *and* ρ_2 *be two fuzzy relations on a universe set* X*, and let* T *be a t-norm. Then, the following statements hold:*

(i) *If* $\rho_1 \bigtriangledown_l^T \rho_2$ *and* $\rho_1 \bigtriangledown_r^T \rho_2$*, then* $\rho_1 \bigtriangledown^T \rho_2$*.*

(ii) *If* $\rho_1 \bigtriangledown^T \rho_2$ *and* ρ_2 *is reflexive, then* $\rho_1 \bigtriangledown_l^T \rho_2$ *and* $\rho_1 \bigtriangledown_r^T \rho_2$*.*

Proof: We prove each part separately.

Proof of (i): Assume $\rho_1 \bigtriangledown_l^T \rho_2$ and $\rho_1 \bigtriangledown_r^T \rho_2$. For all $x,y,z,t \in X$, we have:

$$T\left(|\rho_1^-(x,y)|, |\rho_2^-(x,z)|\right) \le |\rho_1^-(z,y)|,$$
$$T\left(\rho_1^+(x,y), \rho_2^+(x,z)\right) \le \rho_1^+(z,y),$$

and

$$T\left(|\rho_1^-(x,y)|, |\rho_2^-(y,t)|\right) \le |\rho_1^-(x,t)|,$$
$$T\left(\rho_1^+(x,y), \rho_2^+(y,t)\right) \le \rho_1^+(x,t).$$

Applying associativity and monotonicity of the t-norm T, we get:

$$T\left(T(|\rho_1^-(x,y)|, |\rho_2^-(x,z)|), |\rho_2^-(y,t)|\right) \le T(|\rho_1^-(z,y)|, |\rho_2^-(y,t)|) \le |\rho_1^-(z,t)|,$$
$$T\left(T(\rho_1^+(x,y), \rho_2^+(x,z)), \rho_2^+(y,t)\right) \le T(\rho_1^+(z,y), \rho_2^+(y,t)) \le \rho_1^+(z,t).$$

Hence, $\rho_1 \bigtriangledown^T \rho_2$ holds.

Proof of (ii): Assume $\rho_1 \bigtriangledown^T \rho_2$ and that ρ_2 is reflexive. By definition, this means:

$$\rho_2^+(x,x) = 1 \quad \text{and} \quad \rho_2^-(x,x) = 1 \quad \text{for all } x \in X.$$

The assumption $\rho_1 \bigtriangledown^T \rho_2$ implies that for all $x,y,z,t \in X$:

$$T\left(|\rho_1^-(x,y)|, T(|\rho_2^-(x,z)|, |\rho_2^-(y,t)|)\right) \le |\rho_1^-(z,t)|,$$
$$T\left(\rho_1^+(x,y), T(\rho_2^+(x,z), \rho_2^+(y,t))\right) \le \rho_1^+(z,t).$$

Setting $t = y$, and using reflexivity of ρ_2, we get $\rho_2^+(y,y) = 1$ and $\rho_2^-(y,y) = 1$. Thus:

$$T\left(|\rho_1^-(x,y)|, T(|\rho_2^-(x,z)|, 1)\right) = T(|\rho_1^-(x,y)|, |\rho_2^-(x,z)|) \le |\rho_1^-(z,y)|,$$
$$T\left(\rho_1^+(x,y), T(\rho_2^+(x,z), 1)\right) = T(\rho_1^+(x,y), \rho_2^+(x,z)) \le \rho_1^+(z,y).$$

Therefore, $\rho_1 \bigtriangledown_l^T \rho_2$ holds. Similarly, by setting $z = y$, one can show that $\rho_1 \bigtriangledown_r^T \rho_2$ also holds.

This completes the proof.

□

Corollary 9. *If ρ is a reflexive bipolar fuzzy relation on X, then $\rho \bigtriangledown \rho$ if and only if $\rho \bigtriangledown_r \rho$ and $\rho \bigtriangledown_l \rho$.*

Proof (Guidance): Let ρ be a reflexive bipolar fuzzy relation on a universe X. We prove both directions of the equivalence.

($\Rightarrow$) Assume $\rho \bigtriangledown \rho$ holds. To deduce that both $\rho \bigtriangledown_r \rho$ and $\rho \bigtriangledown_l \rho$ also hold:

- Use the assumption $\rho \bigtriangledown \rho$, which involves inequalities with nested t-norms.
- Set specific values (e.g., $y = z$ or $z = y$) to reduce the nested composition to simpler two-element compositions.
- Apply reflexivity of ρ to simplify terms like $\rho(x, x) = 1$.
- Conclude that the simplified expressions correspond exactly to the definitions of $\rho \bigtriangledown_r \rho$ and $\rho \bigtriangledown_l \rho$.

($\Leftarrow$) Now assume both $\rho \bigtriangledown_r \rho$ and $\rho \bigtriangledown_l \rho$ hold. To prove $\rho \bigtriangledown \rho$:

- Recall that the definition of $\rho \bigtriangledown \rho$ involves a three-point composition via nested t-norms.
- Use associativity and monotonicity of the t-norm T to combine the inequalities obtained from the right and left compositions.
- Conclude that the full composition inequality of $\rho \bigtriangledown \rho$ holds by chaining the right and left compositions through T.

Thus, both implications are established, and the corollary follows. □

4.|Compatibility property in terms of bipolar fuzzy relation traces

In this section, we first extend the notion of traces given by Fodor [16] to bipolar fuzzy relation case, and we will provide an interesting characterization of the compatibility of bipolar fuzzy relations in terms of the notions of left and right traces.

4.1.|Left and Right Traces of Bipolar Fuzzy Relations

In this subsection, we extend some basic definitions and results on traces of L-relations, as introduced by Fodor [16], to the bipolar fuzzy case.

Definition 10. *Let $\rho = (\rho^-, \rho^+)$ be a bipolar fuzzy relation on a universe X, and let T be a left-continuous t-norm.*

(i) The left trace of ρ, denoted $\rho^l = \left(\left(\rho^l\right)^-, \left(\rho^l\right)^+\right)$, is a bipolar fuzzy relation on X defined by:

$$\begin{cases} \left(\rho^l\right)^-(x, y) = -\inf_{z \in X} \left(T\left(|\rho^-(z, x)|, |\rho^-(z, y)|\right)\right), \\ \left(\rho^l\right)^+(x, y) = \inf_{z \in X} \left(T\left(\rho^+(z, x), \rho^+(z, y)\right)\right). \end{cases}$$

(ii) The right trace of ρ, denoted $\rho^r = ((\rho^r)^-, (\rho^r)^+)$, is a bipolar fuzzy relation on X defined by:

$$\begin{cases} (\rho^r)^-(x,y) = -\inf_{z\in X} \left(T\left(|\rho^-(y,z)|, |\rho^-(x,z)|\right)\right), \\ (\rho^r)^+(x,y) = \inf_{z\in X} \left(T\left(\rho^+(y,z), \rho^+(x,z)\right)\right). \end{cases}$$

Here's an example illustrating the definition of left and right traces for a bipolar fuzzy relation:

Example 11. *Let the universe $X = \{1, 2, 3\}$, and let $\rho = (\rho^-, \rho^+)$ be a bipolar fuzzy relation on X, where ρ^- and ρ^+ represent the negative and positive parts of the relation, respectively.*

Assume that ρ^- and ρ^+ are defined as follows:

$$\rho^- = \begin{pmatrix} 0 & 0.2 & 0.4 \\ 0.2 & 0 & 0.3 \\ 0.4 & 0.3 & 0 \end{pmatrix}, \quad \rho^+ = \begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.6 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{pmatrix}$$

Here, the value $\rho^-(x,y)$ represents the negative part of the fuzzy relation between x and y, and $\rho^+(x,y)$ represents the positive part of the fuzzy relation between x and y.

We use the product t-norm (denoted T) for this example, where $T(a,b) = a \cdot b$.

Step 1: Calculate the left trace ρ^l

The left trace of ρ, denoted $\rho^l = ((\rho^l)^-, (\rho^l)^+)$, is calculated as follows:

$$(\rho^l)^-(x,y) = -\inf_{z\in X} \left(T\left(|\rho^-(z,x)|, |\rho^-(z,y)|\right)\right)$$
$$(\rho^l)^+(x,y) = \inf_{z\in X} \left(T\left(\rho^+(z,x), \rho^+(z,y)\right)\right)$$

Calculation for $(\rho^l)^-(1,2)$:

$$(\rho^l)^-(1,2) = -\inf\left(T(|\rho^-(1,1)|, |\rho^-(1,2)|), T(|\rho^-(2,1)|, |\rho^-(2,2)|), T(|\rho^-(3,1)|, |\rho^-(3,2)|)\right)$$

Substituting the values of ρ^-:

$$(\rho^l)^-(1,2) = -\inf\left(T(0, 0.2), T(0.2, 0), T(0.4, 0.3)\right)$$

Using the product t-norm $T(a,b) = a \cdot b$:

$$(\rho^l)^-(1,2) = -\inf(0, 0, 0.12) = -0$$

Calculation for $(\rho^l)^+(1,2)$:

$$(\rho^l)^+(1,2) = \inf\left(T(\rho^+(1,1), \rho^+(1,2)), T(\rho^+(2,1), \rho^+(2,2)), T(\rho^+(3,1), \rho^+(3,2))\right)$$

Substituting the values of ρ^+:

$$(\rho^l)^+(1,2) = \inf\left(T(1, 0.6), T(0.6, 1), T(0.8, 0.7)\right)$$

Using the product t-norm $T(a,b) = a \cdot b$:

$$(\rho^l)^+(1,2) = \inf(0.6, 0.6, 0.56) = 0.56$$

Thus, for $x = 1$ and $y = 2$, we have:

$$\left(\rho^l\right)^-(1,2) = 0, \quad \left(\rho^l\right)^+(1,2) = 0.56$$

Step 2: Calculate the right trace ρ^r

The right trace of ρ, denoted $\rho^r = ((\rho^r)^-, (\rho^r)^+)$, is calculated as follows:

$$(\rho^r)^-(x,y) = -\inf_{z\in X}\left(T\left(|\rho^-(y,z)|, |\rho^-(x,z)|\right)\right)$$
$$(\rho^r)^+(x,y) = \inf_{z\in X}\left(T\left(\rho^+(y,z), \rho^+(x,z)\right)\right)$$

Calculation for $(\rho^r)^-(1,2)$:

$$(\rho^r)^-(1,2) = -\inf\left(T(|\rho^-(2,1)|, |\rho^-(1,1)|), T(|\rho^-(2,2)|, |\rho^-(1,2)|), T(|\rho^-(2,3)|, |\rho^-(1,3)|)\right)$$

Substituting the values of ρ^-:

$$(\rho^r)^-(1,2) = -\inf\left(T(0.2,0), T(0,0.2), T(0.3,0.4)\right)$$

Using the product t-norm $T(a,b) = a\cdot b$:

$$(\rho^r)^-(1,2) = -\inf(0,0,0.12) = -0$$

Calculation for $(\rho^r)^+(1,2)$:

$$(\rho^r)^+(1,2) = \inf\left(T(\rho^+(2,1), \rho^+(1,1)), T(\rho^+(2,2), \rho^+(1,2)), T(\rho^+(2,3), \rho^+(1,3))\right)$$

Substituting the values of ρ^+:

$$(\rho^r)^+(1,2) = \inf\left(T(0.6,1), T(1,0.6), T(0.7,0.8)\right)$$

Using the product t-norm $T(a,b) = a\cdot b$:

$$(\rho^r)^+(1,2) = \inf(0.6,0.6,0.56) = 0.56$$

Thus, for $x = 1$ and $y = 2$, we have:

$$(\rho^r)^-(1,2) = 0, \quad (\rho^r)^+(1,2) = 0.56$$

So,

For the bipolar fuzzy relation ρ, the left and right traces are calculated as:

$$\rho^l(1,2) = (0,0.56), \quad \rho^r(1,2) = (0,0.56)$$

This shows how the left and right traces capture the relationship between elements in the universe X, using the product t-norm for the aggregation.

In the following, we explore some interesting connections between the properties of a bipolar fuzzy relation and those of its traces.

Lemma 12. *For a bipolar fuzzy relation ρ on X and a left-continuous t-norm T, the following properties hold:*

(i) *ρ^l and ρ^r are both reflexive;*

(ii) ρ^l *and* ρ^r *are both* T*-transitive;*

(iii) $(\rho^t)^l = (\rho^r)^t$ *and* $(\rho^t)^r = (\rho^l)^t$.

Proof: (i) For all $x \in X$, we have:

$$(\rho^l)^-(x,x) = -\inf_{z\in X} \overrightarrow{T}\left(|\rho^-(z,x)|, |\rho^-(z,x)|\right)$$

$$(\rho^l)^+(x,x) = \inf_{z\in X} \overrightarrow{T}\left(\rho^+(z,x), \rho^+(z,x)\right)$$

For each x, this gives $(\rho^l)^-(x,x) = -1$ and $(\rho^l)^+(x,x) = 1$, so $\rho^l(x,x) = (-1,1)$. This shows that ρ^l is a reflexive bipolar fuzzy relation.

Similarly, we can prove that ρ^r is a reflexive bipolar fuzzy relation.

(ii) To prove that ρ^l is a bipolar fuzzy T-transitive relation, let $x, y, z \in X$. Since $\overrightarrow{T}$ is transitive, we have:

$$\begin{aligned} T\left(|\rho^l)^-(x,y)|, |\rho^l)^-(y,z)|\right) &= T\left(-\inf_{t\in X} \overrightarrow{T}\left(|\rho^-(t,x)|, |\rho^-(t,y)|\right), -\inf_{t\in X} \overrightarrow{T}\left(|\rho^-(t,y)|, |\rho^-(t,z)|\right)\right) \\ &\le \inf_{t\in X} \overrightarrow{T}\left(|\rho^-(t,x)|, |\rho^-(t,z)|\right) = |\rho^l)^-(x,z)| \end{aligned}$$

Similarly,

$$\begin{aligned} T\left((\rho^l)^+(x,y), (\rho^l)^+(y,z)\right) &= \inf_{t\in X} \overrightarrow{T}\left(\rho^+(t,x), \rho^+(t,y)\right), \inf_{t\in X} \overrightarrow{T}\left(\rho^+(t,y), \rho^+(t,z)\right) \\ &\le \inf_{t\in X} \overrightarrow{T}\left(\rho^+(t,x), \rho^+(t,z)\right) = (\rho^l)^+(x,z) \end{aligned}$$

This shows that $T\left(|\rho^l)^-(x,y)|, |\rho^l)^-(y,z)|\right) \le |\rho^l)^-(x,z)|$ and $T\left((\rho^l)^+(x,y), (\rho^l)^+(y,z)\right) \le (\rho^l)^+(x,z)$, implying that ρ^l is a T-transitive bipolar fuzzy relation.

Similarly, we can prove that ρ^r is a bipolar fuzzy T-transitive relation.

(iii) To prove that $(\rho^t)^l = (\rho^r)^t$, let $x, y \in X$. We have:

$$\left((\rho^t)^l\right)^-(x,y) = \inf_{z\in X}\left(-\inf_{t\in X} \overrightarrow{T}\left(|\rho^-(z,x)|, |\rho^-(z,y)|\right)\right) = \left((\rho^r)^t\right)^-(x,y)$$

$$\left((\rho^t)^l\right)^+(x,y) = \inf_{z\in X} \overrightarrow{T}\left(\rho^+(z,x), \rho^+(z,y)\right) = \left((\rho^r)^t\right)^+(x,y)$$

This shows that $(\rho^t)^l = (\rho^r)^t$.

Similarly, we can prove that $(\rho^t)^r = (\rho^l)^t$. □

Theorem 13. *Let* ρ *be a bipolar fuzzy relation on a universe set* X*, then the following statements are equivalent:*

(i) ρ *is reflexive.*

(ii) $\rho^l \subset \rho$.

(iii) $\rho^r \subset \rho$.

Proof: Prove that $(i) \Rightarrow (ii)$: Suppose that ρ is a bipolar fuzzy reflexive relation. For all $x, y \in X$, we have:

$$(\rho^l)^-(x,y) = -\inf_{z\in X} \overrightarrow{T}(|\rho^-(z,x)|, |\rho^-(z,y)|) \ge -\overrightarrow{T}(1, |\rho^-(x,y)|) = -|\rho^-(x,y)| = \rho^-(x,y), \tag{1}$$

and

$$(\rho^l)^+(x,y) = \inf_{z\in X} \overrightarrow{T}(\rho^+(z,x), \rho^+(z,y)) \le \overrightarrow{T}(\rho^+(x,x), \rho^+(x,y)) = \rho^+(x,y). \tag{2}$$

This implies that $\rho^l \subset \rho$.

In the same manner, we prove $(i) \Rightarrow (iii)$.

Prove that $(ii) \Rightarrow (i)$:

Suppose that $\rho^l \subset \rho$, i.e., for all $x, y \in X$, we have:

$$(\rho^l)^-(x,y) \geq \rho^-(x,y), \quad (\rho^l)^+(x,y) \leq \rho^+(x,y).$$

Let $x \in X$. Then:

$$\rho^-(x,x) \leq (\rho^l)^-(x,x) = -\inf_{z \in X} \overrightarrow{T}(|\rho^-(z,x)|, |\rho^-(z,x)|) = -1,$$

and

$$\rho^+(x,x) \geq (\rho^l)^+(x,x) = \inf_{z \in X} \overrightarrow{T}(\rho^+(z,x), \rho^+(z,x)) = 1.$$

Thus, $\rho^-(x,x) = -1$ and $\rho^+(x,x) = 1$, meaning ρ is a bipolar fuzzy reflexive relation. Similarly, we can prove $(iii) \Rightarrow (i)$. □

Theorem 14. *Let ρ be a bipolar fuzzy relation on a universe set X, and let T be a left-continuous t-norm. Then, the following statements are equivalent:*

(i) ρ is T-transitive.

(ii) $\rho \subseteq \rho^l$.

(iii) $\rho \subseteq \rho^r$.

Proof: We prove the equivalence of the three statements.

(i) $\Rightarrow$ (ii): Assume that ρ is T-transitive. For all $x, y \in X$, and for any $z \in X$, transitivity gives:

$$|\rho^-(x,y)| \leq \overrightarrow{T}(|\rho^-(z,x)|, |\rho^-(z,y)|), \quad \rho^+(x,y) \leq \overrightarrow{T}(\rho^+(z,x), \rho^+(z,y)).$$

Taking infima over all $z \in X$, we obtain:

$$|\rho^-(x,y)| \leq \inf_{z \in X} \overrightarrow{T}(|\rho^-(z,x)|, |\rho^-(z,y)|), \quad \rho^+(x,y) \leq \inf_{z \in X} \overrightarrow{T}(\rho^+(z,x), \rho^+(z,y)).$$

Hence,

$$\rho^-(x,y) \geq (\rho^l)^-(x,y), \quad \rho^+(x,y) \leq (\rho^l)^+(x,y),$$

which implies $\rho \subseteq \rho^l$.

(i) $\Rightarrow$ (iii): Similarly, from T-transitivity we also have:

$$|\rho^-(x,y)| \leq \overrightarrow{T}(|\rho^-(z,y)|, |\rho^-(z,x)|), \quad \rho^+(x,y) \leq \overrightarrow{T}(\rho^+(z,y), \rho^+(z,x)).$$

Taking the infimum over all $z \in X$, we deduce:

$$|\rho^-(x,y)| \leq \inf_{z \in X} \overrightarrow{T}(|\rho^-(z,y)|, |\rho^-(z,x)|), \quad \rho^+(x,y) \leq \inf_{z \in X} \overrightarrow{T}(\rho^+(z,y), \rho^+(z,x)),$$

i.e.,

$$\rho^-(x,y) \geq (\rho^r)^-(x,y), \quad \rho^+(x,y) \leq (\rho^r)^+(x,y),$$

which means $\rho \subseteq \rho^r$.

(ii) $\Rightarrow$ (i): Assume that $\rho \subseteq \rho^l$. That is,

$$\rho^-(x,y) \geq (\rho^l)^-(x,y), \quad \rho^+(x,y) \leq (\rho^l)^+(x,y).$$

Recall:

$$(\rho^l)^-(x,y) = -\inf_{z \in X} \overrightarrow{T}(|\rho^-(z,x)|, |\rho^-(z,y)|), \quad (\rho^l)^+(x,y) = \inf_{z \in X} \overrightarrow{T}(\rho^+(z,x), \rho^+(z,y)).$$

Therefore, for all $z \in X$,

$$|\rho^-(x,y)| \leq \overrightarrow{T}(|\rho^-(z,x)|, |\rho^-(z,y)|), \quad \rho^+(x,y) \leq \overrightarrow{T}(\rho^+(z,x), \rho^+(z,y)),$$

which shows that ρ is T-transitive.

(iii) $\Rightarrow$ (i): This direction is proved analogously. Assume $\rho \subseteq \rho^r$, i.e.,

$$\rho^-(x,y) \geq (\rho^r)^-(x,y), \quad \rho^+(x,y) \leq (\rho^r)^+(x,y).$$

Then using the definition of ρ^r, we again get the required inequalities showing that ρ is T-transitive.

Since each condition implies the others, the three statements are equivalent. □

4.2. Some Characterizations of Compatibility in Terms of Traces

In this subsection, we present several characterizations of the compatibility of bipolar fuzzy relations through the concept of traces. We explore the relationships between traces and compatibility, providing a detailed examination of how these notions are interrelated. Following this, we highlight some important results regarding the compatibility of bipolar fuzzy relations, particularly in contexts where the relations possess additional properties, such as reflexivity, symmetry, and T-transitivity.

Theorem 15. *For any two bipolar fuzzy relations $\rho = (\rho^-, \rho^+)$ and $R = (R^-, R^+)$ on a universe X, and a left continuous t-norm T, the following two statements hold:*

(i) $\rho \bigtriangledown_r^T R$ *if and only if* $R \subseteq \rho^l$;

(ii) $\rho \bigtriangledown_l^T R$ *if and only if* $R \subseteq (\rho^r)^t$.

Proof: (i) Suppose that ρ is right compatible with R, i.e., for any $x, y, z \in X$,

$$T\left(\left|\rho^-(x,y)\right|, \left|R^-(y,z)\right|\right) \leq \left|\rho^-(x,z)\right|$$

and

$$T\left(\rho^+(x,y), R^+(y,z)\right) \leq \rho^+(x,z).$$

From the compatibility condition, it follows that:

$$\left|R^-(y,z)\right| \leq \overrightarrow{T}\left(\left|\rho^-(x,y)\right|, \left|\rho^-(x,z)\right|\right) \leq \inf_{x\in X}\left(\left|\rho^-(x,y)\right|, \left|\rho^-(x,z)\right|\right) = \left|\left(\rho^l\right)^-(y,z)\right|$$

and

$$R^+(y,z) \leq \overrightarrow{T}\left(\rho^+(x,y), \rho^+(x,z)\right) \leq \inf_{x\in X} \overrightarrow{T}\left(\rho^+(x,y), \rho^+(x,z)\right) = \left(\rho^l\right)^+(y,z).$$

Hence, we obtain the inequalities:

$$R^-(y,z) \geq \left(\rho^l\right)^-(y,z) \quad \text{and} \quad R^+(y,z) \leq \left(\rho^l\right)^+(y,z), \quad \forall y,z \in X.$$

Thus, $R \subseteq \rho^l$.

Conversely, suppose $R \subseteq \rho^l$. Then for any $y, z \in X$:

$$R^-(y,z) \geq -\inf_{x\in X} \overrightarrow{T}\left(\left|\rho^-(x,y)\right|, \left|\rho^-(x,z)\right|\right) \geq -\overrightarrow{T}\left(\left|\rho^-(x,y)\right|, \left|\rho^-(x,z)\right|\right),$$

and

$$R^+(y,z) \leq \inf_{x\in X} \overrightarrow{T}\left(\rho^+(x,y), \rho^+(x,z)\right) \leq \overrightarrow{T}\left(\rho^+(x,y), \rho^+(x,z)\right).$$

Thus, we have:

$$\left|R^-(y,z)\right| \le \overrightarrow{T}\left(\left|\rho^-(x,y)\right|, \left|\rho^-(x,z)\right|\right),$$

and

$$R^+(y,z) \le \overrightarrow{T}\left(\rho^+(x,y), \rho^+(x,z)\right), \quad \forall x,y,z \in X.$$

It follows that:

$$T\left(\left|\rho^-(x,y)\right|, \left|R^-(y,z)\right|\right) \le \left|\rho^-(x,z)\right|$$

and

$$T\left(\rho^+(x,y), R^+(y,z)\right) \le \rho^+(x,z), \quad \forall x,y,z \in X,$$

i.e., ρ is right compatible with R.

(ii) The assertion follows from Lemma 5, (i), and Lemma 12 (iii). □

Corollary 16. *For any two bipolar fuzzy relations $\rho = (\rho^-, \rho^+)$ on a universe X and T be a left continuous t-norm, the following two statements hold:*

(1) $\rho \bigtriangledown_r^T \rho^l$;

(2) $\rho \bigtriangledown_l^T (\rho^r)^t$.

Proof: We will prove the two statements of the corollary.

1. Proof of $\rho \bigtriangledown_r^T \rho^l$:

We need to show that ρ is right-compatible with ρ^l, i.e., for all $x,y,z \in X$,

$$T\left(\left|\rho^-(x,y)\right|, \left|\rho^-(y,z)\right|\right) \le \left|\rho^-(x,z)\right|$$

and

$$T\left(\rho^+(x,y), \rho^+(y,z)\right) \le \rho^+(x,z).$$

- From the definition of ρ^l, we know that:

$$\rho^l = \left((\rho^l)^-, (\rho^l)^+\right),$$

where for any $y,z \in X$,

$$(\rho^l)^-(y,z) = \inf_{x\in X} \overrightarrow{T}\left(\left|\rho^-(x,y)\right|, \left|\rho^-(x,z)\right|\right)$$

and

$$(\rho^l)^+(y,z) = \inf_{x\in X} \overrightarrow{T}\left(\rho^+(x,y), \rho^+(x,z)\right).$$

- By the definition of compatibility $\bigtriangledown_r^T$, for all $x,y,z \in X$:

$$T\left(\left|\rho^-(x,y)\right|, \left|\rho^-(y,z)\right|\right) \le \left|\rho^-(x,z)\right|$$

and

$$T\left(\rho^+(x,y), \rho^+(y,z)\right) \le \rho^+(x,z),$$

which shows that ρ is right-compatible with ρ^l.

Thus, we have shown that:

$$\rho \bigtriangledown_r^T \rho^l.$$

2. Proof of $\rho \bigtriangledown_l^T (\rho^r)^t$:

We need to show that ρ is left-compatible with $(\rho^r)^t$, i.e., for all $x,y,z \in X$,

$$T\left(\left|\rho^-(x,y)\right|, \left|(\rho^r)^-(y,z)\right|\right) \le \left|\rho^-(x,z)\right|$$

and

$$T\left(\rho^+(x,y), (\rho^r)^+(y,z)\right) \le \rho^+(x,z).$$

- From the definition of $(\rho^r)^t$, we have:

$$(\rho^r)^t = \left((\rho^r)^-, (\rho^r)^+\right),$$

where for any $y, z \in X$,

$$(\rho^r)^-(y,z) = \inf_{x\in X} \overrightarrow{T}\left(\left|\rho^-(x,y)\right|, \left|\rho^-(x,z)\right|\right)$$

and

$$(\rho^r)^+(y,z) = \inf_{x\in X} \overrightarrow{T}\left(\rho^+(x,y), \rho^+(x,z)\right).$$

- By the definition of compatibility $\bigtriangledown_l^T$, for all $x, y, z \in X$:

$$T\left(\left|\rho^-(x,y)\right|, \left|(\rho^r)^-(y,z)\right|\right) \le \left|\rho^-(x,z)\right|$$

and

$$T\left(\rho^+(x,y), (\rho^r)^+(y,z)\right) \le \rho^+(x,z),$$

which shows that ρ is left-compatible with $(\rho^r)^t$.

Thus, we have shown that:

$$\rho \bigtriangledown_l^T (\rho^r)^t.$$

□

Corollary 17. *Let ρ and R be two bipolar fuzzy relations on a universe X and T be a left continuous t-norm. If ρ is reflexive and T-transitive, then it holds that*

(i) $\rho \bigtriangledown_r^T R$ if and only if $R \subseteq \rho$;

(ii) $\rho \bigtriangledown_l^T R$ if and only if $R \subseteq \rho^t$;

(iii) If R is symmetric, then $\rho \bigtriangledown_r^T R$ if and only if $\rho \bigtriangledown_l^T R$.

Proof: Suppose that ρ is reflexive and T-transitive, then it follows from Theorems 13 and **??** that $\rho = \rho^r = \rho^l$. It also holds that $\rho^t = (\rho^r)^t = \left(\rho^l\right)^t$.

The assertion (i) follows from Theorem 15 (i).

The assertion (ii) follows from Theorem 15 (ii).

(iii) If R is symmetric, then $R \subseteq \rho$ if and only if $R \subseteq \rho^t$. The equivalence in (iii) follows from (i) and (ii). □

Corollary 18. *For any two bipolar fuzzy relations ρ and R on a universe X, the following two statements hold*

(i) $\rho \bigtriangledown_r R$ if and only if $\rho^l \bigtriangledown_r R$;

(ii) $\rho \bigtriangledown_l R$ if and only if $\rho^r \bigtriangledown_l R$.

Proof: (i) From Theorem 15, it holds that $\rho \bigtriangledown_r R$ if and only if $R \subseteq \rho^l$. Since ρ^l is reflexive and T-transitive, it follows from Corollary 17 (i) that $\rho^l \bigtriangledown_r R$ if and only if $R \subseteq \rho$. Hence, $\rho \bigtriangledown_r R$ if and only if $\rho^l \bigtriangledown_r R$.

(ii) Follows from Lemma 5 and (i). □

Proposition 19. *Let ρ, R_1 and R_2 be bipolar fuzzy relations on a universe X and T be a left continuous t-norm. If R_1 is reflexive, then it holds that*

(1) $\rho \bigtriangledown_l R_1$, $R_1 \bigtriangledown_l R_2$ and R_2 is symmetric, then $\rho \bigtriangledown_l R_1$;

(2) $\rho \bigtriangledown_l R_1$ and $R_1 \bigtriangledown_r R_2$, then $\rho \bigtriangledown_l R_1$;

(3) $\rho \bigtriangledown_r R_1$, $R_1 \bigtriangledown_l R_2$ and R_2 is symmetric, then $\rho \bigtriangledown_r R_1$;

(4) $\rho \bigtriangledown_r R_1$ and $R_1 \bigtriangledown_r R_2$, then $\rho \bigtriangledown_r R_1$.

Proof: (1) Suppose that $\rho \bigtriangledown_l R_1$, $R_1 \bigtriangledown_l R_2$ and R_2 is symmetric,

it follows that $T\left(\left|\rho^-(x,y)\right|, \left|R_2^-(x,z)\right|\right) = T\left(\left|\rho^-(x,y)\right|, \left|R_2^-(z,x)\right|\right)$ and

$T\left(\rho^+(x,y), R_2^+(x,z)\right) = T\left(\rho^+(x,y), R_2^+(z,x)\right)$, since $R_1 \bigtriangledown_l R_2$ and by Corollary 17 (ii) $R_2 \subseteq R_1^t$, hence

$$\begin{aligned} T\left(\left|\rho^-(x,y)\right|, \left|R_2^-(x,z)\right|\right) &\leq T\left(\left|\rho^-(x,y)\right|, \left|(R_1^t)^-(z,x)\right|\right) \\ &= T\left(\left|\rho^-(x,y)\right|, \left|(R_1)^-(x,z)\right|\right) \\ &\leq \left|\rho^-(z,y)\right| \end{aligned}$$

and

$$\begin{aligned} T\left(\rho^+(x,y), R_2^+(x,z)\right) &\leq T\left(\rho^+(x,y), (R_1^t)^+(z,x)\right) \\ &= T\left(\rho^+(x,y), (R_1)^+(x,z)\right) \\ &\leq \rho^+(z,y). \end{aligned}$$

(2) Suppose that $\rho \bigtriangledown_l R_1$, $R_1 \bigtriangledown_r R_2$, since $R_1 \bigtriangledown_r R_2$ and by Corollary 17 (i) $R_2 \subseteq R_1$, hence

$$\begin{aligned} T\left(\left|\rho^-(x,y)\right|, \left|R_2^-(x,z)\right|\right) &\leq T\left(\left|\rho^-(x,y)\right|, \left|(R_1)^-(x,z)\right|\right) \\ &\leq \left|\rho^-(z,y)\right| \end{aligned}$$

and

$$\begin{aligned} T\left(\rho^+(x,y), R_2^+(x,z)\right) &\leq T\left(\rho^+(x,y), R_1^+(x,z)\right) \\ &\leq \rho^+(z,y). \end{aligned}$$

(3) From Corollary 17 (ii).

(4) From Corollary 17 (i). □

5.|Left and Right-Compatibility of a Bipolar Fuzzy Tolerance (Equivalence) Relation with an Order Relation

Throughout this paper, we use the notation χ_R to refer to the characteristic function of a bipolar relation R.

To avoid any confusion, we emphasize the distinction between the two partial order relations used in this context:

- The symbol $\leqslant$ refers to a partial order relation on the set X, - The symbol $\leq$ refers to the partial order relation on the interval $[0,1]$.

In particular, the characteristic functions for the partial order relation $\leqslant$ are defined as follows:

$$\chi_{\leqslant}^-(x,y) = \begin{cases} -1 & \text{if } x \leqslant y, \\ 0 & \text{if } x \nleqslant y, \end{cases} \quad \text{and} \quad \chi_{\leqslant}^+(x,y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ 0 & \text{if } x \nleqslant y. \end{cases}$$

The following proposition shows the equivalence between all types of compatibility of a bipolar fuzzy tolerance relation and the bipolar characteristic function of an order relation.

Proposition 20. *Let $(X,\leqslant)$ be a poset and E be a bipolar fuzzy tolerance relation on X. Then the following statements are equivalent:*

(i) E is left-compatible with $\chi_{\leqslant}$,

(ii) E is right-compatible with $\chi_{\leqslant}$,

(iii) E is compatible with $\chi_{\leqslant}$.

Proof: (i) ⇒ (ii): Since E is left-compatible with $\chi_{\leqslant}$, by Lemma 5 (i), E^t is right-compatible with $\chi_{\leqslant}$. Since E is a bipolar fuzzy symmetric relation, it follows that E is also right-compatible with $\chi_{\leqslant}$.

(ii) ⇒ (iii): As E is a bipolar fuzzy symmetric relation and is right-compatible with $\chi_{\leqslant}$, we have the following:

$$\begin{aligned} T\left(|E^-(x,y)|, T\left(\left|\chi^-_{\leqslant}(x,z)\right|, \left|\chi^-_{\leqslant}(y,t)\right|\right)\right) &= T\left(T\left(|E^-(y,x)|, \left|\chi^-_{\leqslant}(x,z)\right|\right), \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &\leq T\left(|E^-(y,z)|, \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &= T\left(|E^-(z,y)|, \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &\leq |E^-(z,t)| \end{aligned}$$

and

$$\begin{aligned} T\left(E^+(x,y), T\left(\chi^+_{\leqslant}(x,z), \chi^+_{\leqslant}(y,t)\right)\right) &= T\left(T\left(E^+(y,x), \chi^+_{\leqslant}(x,z)\right), \chi^+_{\leqslant}(y,t)\right) \\ &\leq T\left(E^+(y,z), \chi^+_{\leqslant}(y,t)\right) \\ &= T\left(E^+(z,y), \chi^+_{\leqslant}(y,t)\right) \\ &\leq E^+(z,t) \end{aligned}$$

for any $x, y, z, t \in X$. Hence, E is compatible with $\chi_{\leqslant}$.

(iii) ⇒ (i): This follows from Proposition 8 (ii). □

Corollary 21. *Let $(X, \leqslant)$ be a poset and E be a bipolar fuzzy equivalence (or T-equivalence) relation on X. Then the following statements are equivalent:*

(i) E is left-compatible with $\chi_{\leqslant}$,

(ii) E is right-compatible with $\chi_{\leqslant}$,

(iii) E is compatible with $\chi_{\leqslant}$.

Proof: (i) ⇒ (ii) Since E is left-compatible with $\chi_{\leqslant}$, by Lemma 5 (i), E^t is right-compatible with $\chi_{\leqslant}$. Furthermore, since E is a bipolar fuzzy symmetric relation, it follows that E is also right-compatible with $\chi_{\leqslant}$.

(ii) ⇒ (iii): Since E is a bipolar fuzzy symmetric relation, and E is right-compatible with $\chi_{\leqslant}$, we apply the definition of compatibility to obtain the following inequalities for the left-compatibility condition:

$$\begin{aligned} T\left(|E^-(x,y)|, T\left(\left|\chi^-_{\leqslant}(x,z)\right|, \left|\chi^-_{\leqslant}(y,t)\right|\right)\right) &= T\left(T\left(|E^-(y,x)|, \left|\chi^-_{\leqslant}(x,z)\right|\right), \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &\leq T\left(|E^-(y,z)|, \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &= T\left(|E^-(z,y)|, \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &\leq |E^-(z,t)| \end{aligned}$$

and

$$\begin{aligned} T\left(E^+(x,y), T\left(\chi^+_{\leqslant}(x,z), \chi^+_{\leqslant}(y,t)\right)\right) &= T\left(T\left(E^+(y,x), \chi^+_{\leqslant}(x,z)\right), \chi^+_{\leqslant}(y,t)\right) \\ &\leq T\left(E^+(y,z), \chi^+_{\leqslant}(y,t)\right) \\ &= T\left(E^+(z,y), \chi^+_{\leqslant}(y,t)\right) \\ &\leq E^+(z,t) \end{aligned}$$

for any $x, y, z, t \in X$. Hence, E is compatible with $\chi_{\leqslant}$.

(iii) ⇒ (i): This follows directly from Proposition 8 (ii), since if E is compatible with $\chi_{\leqslant}$, it is both left- and right-compatible with $\chi_{\leqslant}$.

□

The following proposition shows that the left-compatibility (resp.

right-compatibility) of any bipolar fuzzy relation with the bipolar characteristic function of a given strict order relation $<$ is equivalent to the left-compatibility (resp. right-compatibility) with the bipolar characteristic function of the corresponding order relation $\leqslant$.

Corollary 22. *Let $(X, \leqslant)$ be a poset and E be a bipolar fuzzy equivalence (or T-equivalence) relation on X. Then the following statements are equivalent:*

(i) E is left-compatible with $\chi_{\leqslant}$,

(ii) E is right-compatible with $\chi_{\leqslant}$,

(iii) E is compatible with $\chi_{\leqslant}$.

Proof: (i) $\Rightarrow$ (ii): Since E is left-compatible with $\chi_{\leqslant}$, by Lemma 5 (i), E^t is right-compatible with $\chi_{\leqslant}$. Furthermore, since E is a bipolar fuzzy symmetric relation, it follows that E is also right-compatible with $\chi_{\leqslant}$.

(ii) $\Rightarrow$ (iii): Since E is a bipolar fuzzy symmetric relation, and E is right-compatible with $\chi_{\leqslant}$, we apply the definition of compatibility to obtain the following inequalities for the left-compatibility condition:

$$\begin{aligned} T\left(|E^-(x,y)|, T\left(\left|\chi^-_{\leqslant}(x,z)\right|, \left|\chi^-_{\leqslant}(y,t)\right|\right)\right) &= T\left(T\left(|E^-(y,x)|, \left|\chi^-_{\leqslant}(x,z)\right|\right), \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &\leq T\left(|E^-(y,z)|, \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &= T\left(|E^-(z,y)|, \left|\chi^-_{\leqslant}(y,t)\right|\right) \\ &\leq |E^-(z,t)| \end{aligned}$$

and

$$\begin{aligned} T\left(E^+(x,y), T\left(\chi^+_{\leqslant}(x,z), \chi^+_{\leqslant}(y,t)\right)\right) &= T\left(T\left(E^+(y,x), \chi^+_{\leqslant}(x,z)\right), \chi^+_{\leqslant}(y,t)\right) \\ &\leq T\left(E^+(y,z), \chi^+_{\leqslant}(y,t)\right) \\ &= T\left(E^+(z,y), \chi^+_{\leqslant}(y,t)\right) \\ &\leq E^+(z,t) \end{aligned}$$

for any $x, y, z, t \in X$. Hence, E is compatible with $\chi_{\leqslant}$.

(iii) $\Rightarrow$ (i): This follows directly from Proposition 8 (ii), since if E is compatible with $\chi_{\leqslant}$, it is both left- and right-compatible with $\chi_{\leqslant}$.

□

Theorem 23. *Let $(X, \leqslant)$ be a poset, $\rho = (\rho^-, \rho^+)$ be a bipolar fuzzy relation on X, and T be a t-norm. Then the following statements hold:*

(i) $\chi_{\leqslant} \bigtriangledown^T_l \rho$ if and only if $\rho \subseteq \chi^t_{\leqslant}$,

(ii) $\chi_{\leqslant} \bigtriangledown^T_r \rho$ if and only if $\rho \subseteq \chi_{\leqslant}$,

(iii) $\chi_{\leqslant} \bigtriangledown^T \rho$ if and only if $\rho \subseteq \chi_{\leqslant} \cap \chi^t_{\leqslant}$.

Proof: (i) Suppose that $\chi_{\leqslant}$ is left-compatible with ρ, i.e., for any $x, y, z \in X$,

$$T\left(\left|\chi^-_{\leqslant}(x,y)\right|, |\rho^-(x,z)|\right) \leq \left|\chi^-_{\leqslant}(z,y)\right|,$$

and

$$T\left(\chi^+_{\leqslant}(x,y), \rho^+(x,z)\right) \leq \chi^+_{\leqslant}(z,y),$$

Since $\chi_{\leqslant}$ is reflexive, we also have:

$$T\left(\left|\chi^-_{\leqslant}(x,x)\right|, \left|\rho^-(x,y)\right|\right) \leq \left|\chi^-_{\leqslant}(y,x)\right|,$$

and

$$T\left(\chi^+_{\leqslant}(x,x), \rho^+(x,y)\right) \leq \chi^+_{\leqslant}(y,x),$$

which implies:

$$\rho^-(x,y) \geq \chi^-_{\leqslant}(y,x) \quad \text{and} \quad \rho^+(x,y) \leq \chi^+_{\leqslant}(y,x).$$

Thus, we have:

$$\rho^-(x,y) \geq (\chi^t_{\leqslant})^-(x,y) \quad \text{and} \quad \rho^+(x,y) \leq (\chi^t_{\leqslant})^+(x,y),$$

which implies that $\rho \subseteq \chi^t_{\leqslant}$.

Conversely, if $\rho \subseteq \chi^t_{\leqslant}$, then we have the following for any $x, y, z \in X$:

$$T\left(\left|\chi^-_{\leqslant}(x,y)\right|, \left|\rho^-(x,z)\right|\right) \leq T\left(\left|\chi^-_{\leqslant}(x,y)\right|, \left|(\chi^t_{\leqslant})^-(x,z)\right|\right) = T\left(\left|\chi^-_{\leqslant}(x,y)\right|, \left|\chi^-_{\leqslant}(z,x)\right|\right) \leq \left|\chi^-_{\leqslant}(z,y)\right|,$$

and

$$T\left(\chi^+_{\leqslant}(x,y), \rho^+(x,z)\right) \leq T\left(\chi^+_{\leqslant}(x,y), (\chi^t_{\leqslant})^+(x,z)\right) = T\left(\chi^+_{\leqslant}(x,y), \chi^+_{\leqslant}(z,x)\right) \leq \chi^+_{\leqslant}(z,y).$$

Hence, $\chi_{\leqslant}$ is left-compatible with ρ.

(ii) The assertion for right-compatibility follows from Lemma 5 and the same reasoning as in part (i).

(iii) The assertion follows from the fact that for both left- and right-compatibility, ρ must satisfy the intersection condition: $\rho \subseteq \chi_{\leqslant} \cap \chi^t_{\leqslant}$, as it must satisfy both conditions simultaneously.

□

6.|Conclusion

In this paper, we have extended Bělohlávek's notions of compatibility to the bipolar fuzzy context. Specifically, we introduced the concepts of left and right compatibility for bipolar fuzzy relations. Additionally, we adapted Fodor's notions of traces to the bipolar fuzzy framework, exploring interesting connections between the properties of a bipolar fuzzy relation and its traces. Through this, we obtained some key characterizations of the compatibility of bipolar fuzzy relations in terms of these trace notions.

Furthermore, we provided a comprehensive characterization of bipolar fuzzy tolerance (or T-equivalence) relations that are compatible with a given bipolar characteristic function of (strict) order relations, shedding light on their intricate relationships.

These results contribute to the broader understanding of fuzzy relations and their compatibility properties, opening the door for further exploration in both theoretical and practical applications in fuzzy logic and related fields.

|Declarations

Author Contributions: Professor Brahim Ziane acted as the principal investigator and lead contributor for this research. He was responsible for the conceptual development, methodological design, formal analysis, theoretical investigation, manuscript drafting, and overall supervision of the work. Dr. Aissa Bouad and Dr. Abdelaziz Amroune contributed to the refinement of the methodology, validation of results, and critical review of the manuscript. The late Professor Ali Oumhani, who passed away on February 12, 2024, originated the central idea of the study and provided the initial conceptual framework, literature synthesis, and early-stage supervision. His foundational contribution is deeply acknowledged, and this paper is respectfully dedicated to his memory.

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