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Fixed Point in Non-Archimedeant T -Banach Spaces

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Abstract

In this paper, we introduce the concept of T-normed space in a non-Archimedean space, which is called a non-Archimedean T-normed space and give some properties. After that we prove a common fixed point theorem in a complete non-Archimedean T-normed space for two R-weakly commuting mappings.

Keywords

non-Archimedean Banach space; non-Archimedean T-normed space; fixed point; R-weakly commuting mappings.

2020 Mathematics Subject Classification

Primary 47H10 · Secondary 47A10, 47A55 ·

1. Introduction

In real and complex functional analysis, the theory of fixed points is one of the most useful and widely applied areas. It has been extensively developed and extended in various directions. For instance, in the context of b-metric spaces, see [1, 3, 4, 11]; for generalized metric spaces, see [2]; and for b-generalized metric spaces, see [16].

In a related development, several authors have extended fixed point theory to non-Archimedean fields. For example, Petalas and Vidalis [15] showed that every contractive mapping on a spherically complete non-Archimedean normed space has a unique fixed point.

Non-Archimedean analysis, which is based on a non-Archimedean valuation—such as the p-adic valuation—was introduced by Hensel as early as 1908 (see [10]). Since the 1940s, non-Archimedean analysis has been studied from various perspectives. In 1943, Monna [12] introduced the concept of non-Archimedean normed vector spaces. Notable works in this direction include [6, 5, 8, 9].

The concept of R-weakly commuting mappings between metric spaces was first introduced by Pant [13] in 1994. Later, Vasuki [20] established common fixed point theorems for such mappings in fuzzy metric spaces. More recently, in 2007, V. Pant and R. P. Pant [14] introduced the notion of R-weakly commuting mappings of type (Ag) in fuzzy metric spaces. In 2012, Sedghi, Shobe, and Firouzian [18] generalized the concept of ultrametric spaces by introducing what they called a T-metric space. They also proved a common fixed point theorem for two R-weakly commuting mappings in complete T-metric spaces.

In this paper, we extend the concept of non-Archimedean normed spaces by introducing a new structure called a *non-Archimedean T-normed space*. We establish several basic properties of this new concept. Furthermore, we prove a common fixed point theorem for two R-weakly commuting mappings in complete non-Archimedean T-normed spaces.

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The rest of the paper is organized as follows:

- **Section 2** introduces the necessary notations and recalls basic concepts from the theory of non-Archimedean Banach spaces and T-metric spaces. We also define the notion of non-Archimedean T-normed spaces and discuss some of their properties.
- **Section 3** recalls some foundational results on common fixed point theorems for two R-weakly commuting mappings in complete T-metric spaces (for more details, see [18]). We then extend the concept of R-weakly commuting mappings to the setting of non-Archimedean T-normed spaces. Finally, we present and prove a common fixed point theorem for two such mappings in this new setting. The proof closely follows the structure of the corresponding result for T-metric spaces given in [18, Theorem 3.4].

2. Non-Archimedean T-normed spaces

Before introducing the notion of a generalized non-Archimedean normed space (or non-Archimedean T-normed space), we recall some preliminaries concerning non-Archimedean spaces, including the concept of a non-Archimedean valuation on a general field $\mathbb{K}$ and the definition of a non-Archimedean Banach space (see [6]).

Definition 2.1. A valuation on the field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \longrightarrow \mathbb{R}$ satisfying:
(i) $|x| \geq 0$ for all $x \in \mathbb{K}$, with equality if and only if $x = 0$;

(ii) $|xy| = |x||y|$ for all x, $y \in \mathbb{K}$;

(iii) for some real number $c \geq 1$ and all $x \in \mathbb{K}$, if $|x| \leq 1$, then $|x+1| \leq c$.

Definition 2.2. (i) We say that two valuations $|\cdot|_1$ and $|\cdot|_2$ on the field $\mathbb{K}$ are equivalent if there exists a positive real number λ such that $|\cdot|_2 = |\cdot|_1^{\lambda}$.

(ii) A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the triangle inequality if, for all $x, y \in \mathbb{K}$,

$$|x+y| \leq |x| + |y|.$$

(iii) A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the strong triangle inequality if, for all $x, y \in \mathbb{K}$,

$$|x+y| \leq \max\Big\{|x|, |y|\Big\}.$$

Remark 2.3. (i) A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the triangle inequality if and only if $c = 2$ (see [7, Proposition 1.6]).

(ii) Every valuation on $\mathbb{K}$ is equivalent to one that satisfies the triangle inequality.

(iii) A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the strong triangle inequality if and only if $c = 1$ (see [7, Proposition 1.13]).

According to Remark 2.3 (ii), we may and will assume that the valuation $|\cdot|$ on $\mathbb{K}$ satisfies the triangle inequality. Then, it induces a natural distance function

$$\begin{array}{rcl} d : \mathbb{K} \times \mathbb{K} & \longrightarrow & \mathbb{R}_+ \\ (x, y) & \longmapsto & |x - y|, \end{array}$$

giving it a structure of a metric space $(\mathbb{K}, d)$ (see [7, Proposition 1.17]). As it is known, there are two kinds of valuation, one is the Archimedean valuation, as in the cases of $\mathbb{C}$ and $\mathbb{R}$, and the other is the non-Archimedean valuation.

Definition 2.4. A valuation on $\mathbb{K}$ satisfying the strong triangle inequality is called a non-Archimedean valuation.

Definition 2.5. Let $|\cdot|$ be a non-Archimedean valuation on $\mathbb{K}$.
(i) A sequence (x_n) of $\mathbb{K}$ is said to converge to $x \in \mathbb{K}$ if $|x_n - x| \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq n_0$.
(ii) A sequence (x_n) of $\mathbb{K}$ is called a Cauchy sequence if, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ whenever $n \geq m \geq n_0$.

In the following, $(\mathbb{K}, |\cdot|)$ is a complete non-Archimedean valued field. That is, every Cauchy sequence converges. Now, we define the non-Archimedean norm.

Definition 2.6. Let X be a vector space over $(\mathbb{K}, |\cdot|)$. A non-Archimedean norm on X is a function $\|\cdot\| : X \to \mathbb{R}_+$ satisfying the following:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\lambda x\| = |\lambda|\ \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$;
(iii) $\|x+y\| \leq \max\Big\{\|x\|, \|y\|\Big\}$ for all $x, y \in X$.

Remark 2.7. The non-Archimedean normed space is different from the classical normed space (see [19]) in the sense that the usual requirement on the norm function

$$\|x+y\| \leq \|x\| + \|y\|, \text{ for all } x, y \in X,$$

is replaced by $\|x+y\| \leq \max\Big\{\|x\|, \|y\|\Big\}$ for all $x, y \in X$.

Remark 2.8. (i) Let $\Big(X, \|\cdot\|\Big)$ be a non-Archimedean normed space. For x, $y \in X$, we have

$$\|x+y\| = \max\Big\{\|x\|, \|y\|\Big\}, \quad \text{if} \quad \|x\| \neq \|y\|$$

(see [7, Proposition 2.9]).

(ii) The non-Archimedean valuation on $\mathbb{K}$ itself is a non-Archimedean norm (see [7, Example 2.3]).

Definition 2.9. A non-Archimedean Banach space $(X, \|\cdot\|)$ is a non-Archimedean normed space, which is complete with respect to the natural metric induced by the norm $d(x, y) = \|x - y\|$ for all $x, y \in X$.

The following definition was introduced by Sedghi et al. in [18] which is a probable modification of the definition of ordinary metric.

Definition 2.10. Let X be a nonempty set. A function $T : X \times X \to \mathbb{R}$ is called a generalized metric (or T -metric) on X if it verifies the following conditions, for all x, y, z in X,
(1) $T(x, y) \geq 0$;
(2) $T(x, y) = 0$ if and only if $x = y$;
(3) $T(x, y) = T(y, x)$ (symmetry);
(4) $T(x, z) \leq T(x, z)\Diamond T(y, x)$.
The generalized metric (or T-metric) space is denoted by $(X, T, \Diamond)$, where $\Diamond : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and associative and commutative binary operation satisfying:
(i) $x\Diamond 0 = x$ for all x in $[0, \infty)$;
(ii) $x\Diamond y \leq z\Diamond t$ whenever $x \leq z$ and $y \leq t$ for all $x, y, z, t \in [0, \infty)$.

Remark 2.11. Every ordinary metric d is a T-metric with $x\Diamond y = x + y$.
(2) Every ultra metric d is a T-metric with $x\Diamond y = \max\{x, y\}$.

Now, we give the definition of generalized norm (or T-norm) on X.

Definition 2.12. Let X be a vector space (over $\mathbb{K}$ equal to $\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\|_T : X \to \mathbb{R}_+$ is called a generalized norm (or T-norm) on X if, for all x, y in X, we have
(1)$\|x\|_T = 0 \Longleftrightarrow x = 0$,
(2)$\|\alpha x\|_T = \alpha\|x\|_T$,
(3)$\|x + y\|_T \leq \|x\|_T\Diamond\|y\|_T$.
The 3-tuple $(X, \|.\|_T, \Diamond)$ is called a generalized normed (or T-normed) space.

Remark 2.13. (1) Every usual norm is a T-norm with $x\Diamond y = x + y$.
(2) The metric induced by the T-norm, i.e., $T(x, y) = \|x - y\|_T)$ is T-metric.

Definition 2.14. Let X be a vector space over $(\mathbb{K}, |\cdot|)$. A generalized non-Archimedean norm (or non-Archimedean T -norm) on X is a function $\|\cdot\|_T : X \to \mathbb{R}_+$ verifying the following: for all x, y in X and α in $\mathbb{K}$,
(i) $\|x\|_T = 0 \Leftrightarrow x = 0$;
(ii) $\|\alpha x\|_T = |\alpha|\, \|x\|_T$;
(iii) $\|x + y\|_T \leq \|x\|_T \Diamond \|y\|_T$.
The non-Archimedean T-normed space is denoted by $(X, \|.\|_T, \Diamond)$.

Remark 2.15. Every non-Archimedean normed space is a non-Archimedean T-normed space with $x\Diamond y = \max\{x, y\}$.

Definition 2.16. Let $\big(X, \|.\|_T, \Diamond\big)$ be a non-Archimedean T-normed space, $r > 0$ and $A \subset X$.
(1) The open ball centered at a and radius r is defined by
$B_T(a, x) = \{x \in X \text{ such that } \| a - x \|_T < r\}$;
(2) A subset A is called an open subset of X if, for all $a \in A$, there is $r > 0$ such that $B_T(a, x) \subset A$;
(3) A subset A is said to be T-bounded if there exists $r > 0$ such that $\| x - y \|_T < r$ for all x, y in A;
(4) A sequence $(x_n)_n$ in X is T-convergent to x if $\| x_n - x \|_T \to 0$ where $n \to \infty$ and we write $\lim x_n = x$. That is, for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, t $\| x_n - x \|_T < \epsilon$;
(5) A sequence $(x_n)_n$ in X is T-Cauchy if, for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $\| x_n - x_m \|_T < \epsilon$;
(6) A non-Archimedean T-normed space $\big(X, \|.\|_T, \Diamond\big)$ is said to be complete if every T-Cauchy sequence in X is convergent and we call it a non-Archimedean T-Banach space.

The proof of the following proposition is similar to the case of T-metric space (see [18, Lemme 2.10]).

Proposition 2.17. *A non-Archimedean T -normed space $(X, \|.\|_T, \Diamond)$ is a Hausdorff space.*

Proof. For all $x \neq y$ in X, if we put $\|x - y\|_T = r$, then by [18, Lemme 2.10], for $0 < \delta < r$, there is $0 < \delta' < r$ such that $\delta\Diamond\delta' < r$. Now, it is enough to show that $B(x, \delta) \cap B(y, \delta') = \Phi$. For every z in $B(x, \delta) \cap B(y, \delta')$, by the condition (iii) in Definition 2.14, one finds $r = \|x - y\|_T \leq \|x - z\|_T\Diamond\|z - y\|_T \leq \delta\Diamond\delta' < r$. This is a contradiction. Thus X is a Hausdorff space. □

Proposition 2.18. *Every convergent sequence (x_n) in a non-Archimedean T-normed space $(X, \|.\|_T, \Diamond)$ is a Cauchy sequence.*

Proof. The proof is the same technique as in the proof of [18, Lemme 2.12] by replacing T-metric by the metric induced with the non-Archimedean T-norm. □

3. Fixed point theorems in non-Archimedean T-normed spaces

Now, we recall some basic concepts concerning common fixed point theorems for two R-weakly commuting mappings in complete T -metric spaces satisfying a new contractive type condition. For more detail, see [18].

Definition 3.1. [17] Mappings f and g from a T-metric space (X, T) into itself are said to be R-weakly commuting if there exists some positive real number R such that $T(fg(x), gf(x)) \leq RT(f(x), g(x))$ for all x in X.

The following theorem shows that the common fixed point theorems for two R-weakly commuting mappings in a complete T-metric space, where $\Diamond$ is a binary operation on $[0, \infty) \times [0, \infty)$ such that
(i) $\alpha \cdot (x \Diamond y) = \alpha \cdot x \Diamond \alpha \cdot y$;
(ii) there exists $h \geq 0$ such that $\underbrace{1 \Diamond 1 \cdot \Diamond 1}_{n} \leq n^h$.

Theorem 3.2. *[18] Let (X, T) be a complete T-metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*
(1) $f(X) \subseteq g(X)$;
(2) f or g is continuous;
(3) $T(f(x), f(y)) \leq aT(g(x), g(y)) \Diamond bT(f(x), g(x)) \Diamond cT(f(y), g(y))$ for all $a, b, c \in [0, \infty)$ with $a \Diamond b \Diamond c < 1$.
Then f and g have a unique common fixed point.

Now, we introduce the notion of R-weakly commuting mappings in non-Archimedean T-normed spaces.

Definition 3.3. The mappings f and g from a non-Archimedean T-normed spaces $\left(X, \|.\|_T, \Diamond\right)$ into itself are said to be an R-weakly commuting mapping if there exists some positive real number R such that

$$\|fg(x) - gf(x)\|_T \leq R \|f(x) - g(x)\|_T$$

for all x in X.

Now, we introduce a theorem, corresponding to Theorem 3.2, for a non-Archimedean T-normed space. Its proof is inspired by T-metric case (see [18, Theorem 3.4]). We write the proof for the aim of completeness. Also, we need the following lemma in which the proof is similar to [18, Lemma 3.3] and will be omitted.

Lemma 3.4. *Let $(X, \|.\|_T, \Diamond)$ be a T-normed space. If a sequence $\{x_n\}$ in X such that*

$$\|x_n - x_{n+1}\|_T \leq k\|x_n - x_{n-1}\|_T$$

for all $n \in \mathbb{N}$ and for every $0 \leq k \leq 1$, the sequence $\{x_n\}$ is a Cauchy sequence.

Theorem 3.5. *Let f and g be R-weakly commuting self-mappings of a complete non-Archimedean T-normed space $(X, \|.\|_T, \Diamond)$ such that the following conditions are satisfied:*
(i) $f(X) \subseteq g(X)$;
(ii) f or g is continuous;
(iii) $\|f(x) - f(y)\|_T \leq a\|g(x) - g(y)\|_T \Diamond b\|f(x) - g(x)\|_T \Diamond c\|f(y) - g(y)\|_T$ for all $a, b, c \in [0, \infty)$ with $a \Diamond b \Diamond c < 1$.
Then f and g have a unique common fixed point.

Proof. For an arbitrary element x_0 in X, $f(x_0) \in f(X)$. In fact, if $f(X) \subseteq g(X)$, then there is $x_1 \in X$ such that $f(x_0) = g(x_1)$. By induction, we define a sequence $(x_n)_n$ such that $f(x_n) = g(x_{n+1})$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} &\|f(x_n) - f(x_{n+1})\|_T \\ &\leq a\|g(x_n) - g(x_{n+1})\|_T \Diamond b\|f(x_n) - g(x_n)\|_T \Diamond c\|f(x_{n+1}) - g(x_{n+1}))\|_T \\ &= a\|f(x_{n-1}) - f(x_n)\|_T \Diamond b\|f(x_n) - f(x_{n-1})\|_T \Diamond c\|f(x_{n+1}) - f(x_n)\|_T. \end{aligned} \tag{1}$$

Suppose that $\|f(x_n) - f(x_{n+1})\|_T \geq \|f(x_{n-1}) - f(x_n)\|_T$ and by using (1), we obtain

$$\begin{aligned} \|f(x_n) - f(x_{n+1})\|_T &< a\|f(x_{n-1}) - f(x_n)\|_T \Diamond b\|f(x_n) - f(x_{n-1})\|_T \Diamond |f(x_{n+1}) - f(x_n)\|_T \\ &= \|f(x_n) - f(x_{n+1})\|_T (a \Diamond b \Diamond c) \\ &= \|f(x_n) - f(x_{n+1})\|_T, \end{aligned}$$

which is a contradiction to the assumption. Hence $\|f(x_n) - f(x_{n+1})\|_T \leq \|f(x_{n-1}) - f(x_n)\|_T$. We show that $(f(x_n)_n$ is a Cauchy sequence. (1) and (2) yield $\|f(x_n) - f(x_{n+1})\|_T \leq (a \Diamond b \Diamond c)\|f(x_{n-1}) - f(x_n)\|_T$. By Lemma (3.4), the sequence $(f(x_n)_n$ is Cauchy. Since X is complete, $(f(x_n)_n \to z \in X$, where $n \to \infty$. So $(g(x_n)_n$ converges to $z \in X$. Assume that f is continuous. Then $lim_n f(f(x_n)) = f(z)$ and $lim_n f(g(x_n)) = f(z)$. In fact, since f and g are R-weakly commuting, we have $\|fg(x_n) - gf(x_n)\|_T \leq R\|f(x_n) - g(x_n)\|_T$ and by letting $n \to \infty$ we get $lim_n gf(x_n)) = f(z)$. It remains to show that $f(z) = z$. Set $f(z) \neq z$. Then by (iii), we have

$$\|f(x_n) - ff(x_n)\|_T \leq a\|g(x_n) - gf(x_n)\|_T \Diamond b\|f(x_n) - g(x_n)\|_T \Diamond c\|ff(x_n) - gf(x_n)\|$$

and taking the limit as $n \to \infty$, we find

$$\|z - ff(z)\|_T \leq a\|z - f(z)\|_T \Diamond b\|z - z)\|_T \Diamond c\|f(z) - f(z)\|_T = a\|z - f(z)\|_T < \|z - f(z)\|_T,$$

which is a contradiction. Thus $z = f(z)$. Since $f(X) \subseteq g(X)$, there exists $z_1 \in X$ such that $z = f(z) = g(z)$.

Now, we prove that z is a common fixed point of f and g. Since

$$\begin{aligned}&lim_{n\to\infty}\|ff(x_n) - f(z_1)\|_T\\&\leq lim_{n\to\infty}a\|gf(x_n) - g(z_1)\|_T\lozenge b\|ff(x_n) - gf(x_n)\|_T\lozenge c\|f(z_1) - g(z_1)\|_T,\end{aligned}$$

$$\begin{aligned}&lim_{n\to\infty}\|f(z) - f(z_1)\|_T\\&\leq lim_{n\to\infty}a\|g(z) - g(z_1)\|_T\lozenge b\|f(z) - f(z)\|_T\lozenge c\|z_1 - g(z_1)\|_T,\\&= 0,\end{aligned}$$

which implies that $f(z) = f(z_1)$ and $z = f(z) = f(z_1) = g(z_1)$. Thus

$$\|f(z) - g(z)\| = \|fg(z_1) - gf(z_1)\| \leq R\|f(z_1) - gf(z_1)\| = 0,$$

which implies that $f(z) = g(z)$. To prove the uniqueness of common fixed point of f and g, we consider that there is another common fixed point $t \neq z$. Then

$$\begin{aligned}\|z - t\| = \|f(z) - f(t)\| &\leq a\|g(z) - g(t)\|_T\lozenge b\|f(z) - g(z)\|_T\lozenge c\|f(t) - g(t)\|\\&= a\|z - t\|_T < \|z - t\|_T,\end{aligned}$$

which is a contradiction, and thus $t = z$. This proves that z is a unique common fixed point of f and g. □

The following corollary follows from Theorem 3.5.

Corollary 3.6. *Let f and g be R-weakly commuting self-mappings in a complete non-Archimedean space X such that the following conditions hold:*
(i) $f(X) \subseteq g(X)$;
(ii) f or g is continuous;
(iii) $\|f(x) - f(y)\| \leq \max[\|g(x) - g(y)\|_T, b\|f(x) - g(x)\|_T, c\|f(y) - g(y)\|_T]$ for all $a, b, c \in [0,\infty)$ with $a\lozenge b\lozenge c < 1$.
Then f and g have a unique common fixed point.

We recall that a non-Archimedean normed space $(X, \|\cdot\|)$ is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection. Then the following theorem is a special case of Theorem 3.5.

Theorem 3.7. *[15] Let X be a non-Archimedean spherically complete normed space. If f is a contractive mapping, then T has a unique fixed point.*

Proof. It is enough to take $T(x, y) = \|x - y\|$, $x\lozenge y = max[x, y]$ and $g = I$, the identity mapping, and by Theorem 3.5, the proof is complete. □

Example 3.8. An example satisfying all conditions of Theorem 3.5 is constructed as follows.

Proof. Let the complete non-Archimedean T-normed space be $(X, \|\cdot\|_T, \lozenge)$ where:

- $X = \{0, 1\}$,
- $\|x - y\|_T = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ (the discrete metric),
- $a\lozenge b = \min(a, b)$ (the minimum t-norm).

Define the self-mappings $f, g : X \to X$ by:

$$f(0) = 0, \quad f(1) = 0 \qquad \text{and} \qquad g(0) = 0, \quad g(1) = 1.$$

Let the constants for the contractive condition be $a = b = c = 0$. We now verify all hypotheses of Theorem 3.5:

Condition (i): $f(X) = \{0\}$ and $g(X) = \{0, 1\}$. Thus, $f(X) \subseteq g(X)$.

Condition (ii): In the discrete topology, every function is continuous. Hence, both f and g are continuous.

R-weakly commuting: For any $x \in X$:

$$\|f(g(x)) - g(f(x))\|_T = \|0 - 0\|_T = 0 \quad \text{and}$$

$$\|f(x) - g(x)\|_T = \begin{cases} \|0 - 0\|_T = 0, & \text{if } x = 0 \\ \|0 - 1\|_T = 1, & \text{if } x = 1 \end{cases}$$

Thus, $\|f(g(x)) - g(f(x))\|_T = 0 \leq R \cdot \|f(x) - g(x)\|_T$ holds for any $R \geq 0$ (e.g., $R = 1$). Therefore, f and g are R-weakly commuting.

Condition (iii): We must show:

$$\|f(x) - f(y)\|_T \leq \min\left(a\|g(x) - g(y)\|_T,\ b\|f(x) - g(x)\|_T,\ c\|f(y) - g(y)\|_T\right).$$

Since $a = b = c = 0$, the right-hand side is 0 for all $x, y \in X$. The left-hand side is $\|f(x) - f(y)\|_T = \|0 - 0\|_T = 0$. Therefore, $0 \leq 0$ holds for all x, y. Note that $a\lozenge b\lozenge c = \min(0, 0, 0) = 0 < 1$.

Since all conditions are satisfied, by Theorem 3.5, f and g have a unique common fixed point. By inspection, $f(0) = 0$ and $g(0) = 0$, so $z = 0$ is the unique common fixed point. □

4. Conclusion

We introduced the concept of non-Archimedean T-normed space and gave some properties. Furthermore, we proved a common fixed point theorem in a complete non-Archimedean T-normed space for two R-weakly commuting mappings. Finally, we provided an example.

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Declarations

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