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Integral transformations in L^p spaces

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Contents

Notations

• $(e_1, e_2, ..., e_n)$ is the canonical basis in $\mathbb{R}^n$.
• $x \cdot y = x_1y_1 + ... + x_ny_n$ is the scalar product in $\mathbb{R}^n$.
• For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm in $\mathbb{R}^n$.
• $a.\ e.$ designates almost everywhere
• $(f \star g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$ is the product of the convolution of the functions f and g.
• If $f : \mathbb{R}^n \longrightarrow \mathbb{C}$, the support of f is denoted by $supp\ f$.
• $\mathcal{D}(\mathbb{R}^n)$ is the space of functions $\mathcal{C}^\infty(\mathbb{R}^n)$ with compact support, $\mathcal{D}'(\mathbb{R}^n)$ is the dual space of $\mathcal{D}(\mathbb{R}^n)$, is also called the space of distributions on $\mathbb{R}^n$.
• $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, consisting of rapidly decreasing $\mathcal{C}^\infty(\mathbb{R}^n)$ functions on $\mathbb{R}^n$, the dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.
• If $f \in L^1(\mathbb{R})$, then its Fourier transform is:

$$\mathcal{F}(f(x))(\xi) = \int_{\mathbb{R}} \exp(-2\pi i x.\xi) f(x)dx$$

and its inverse Fourier transform is:

$$\mathcal{F}^{-1}(\widehat{f}(\xi))(x) = \int_{\mathbb{R}} \exp(2\pi i x.\xi)\widehat{f}(\xi)d\xi$$

• q is the conjugate exponent of p, $\frac{1}{p} + \frac{1}{q} = 1$ where $p \in [1, +\infty]$.
• Let $a \in \mathbb{R}^n$, τ_a is the translation operator defined by $\tau_a f(\cdot) = f(\cdot - a)$.
• $L^p(\mathbb{R}^n)$ is the space of measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{\frac{1}{p}} < \infty.$$

• ℓ^q is the space of sequences $(a_k)_k$ such that $\|(a_k)\|_{\ell^q} = (\sum_{k=0}^{\infty} |a_k|^q)^{\frac{1}{q}} < \infty$.
• C_L denotes the class of causal functions that are piecewise continuous and of exponential order.
• $F(s)$ denotes the Laplace transform of the function f.

CHAPTER 1

L^p SPACES

Lebesgue spaces are Banach spaces, i.e., complete normed vector spaces, whose definition and study require the theory of integration.
In this entire chapter, we fix once and for all a measure space. $(X, \mathcal{M}, \mu)$.

1.1 Convex functions and inequalities

Definition 1.1. A function f defined on the open interval $]a, b[$ and taking values in $\mathbb{R}$ is said to be convex if, For all x, y and λ such that

$$a < x < b \quad , \quad a < y < b \quad and \quad 0 \leq \lambda \leq 1,$$

we have

$$f\left((1-\lambda)\, x + \lambda y\right) \leq (1-\lambda)\, f(x) + \lambda f(y).$$

If $-f$ is convex, f is called concave.

Remarks 1.1. 1) If f is convex on $]a, b[$ and if x_1, x_2 *and* x_3 are such that $a < x_1 < x_2 < x_3 < b$, then we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

2) If f is convex, and if f' exists on $]a, b[$ it follows that for x_1 *and* x_2 such that $a < x_1 < x_2 < b$, we have

$$f'(x_1) \leq f'(x_2).$$

3) If f is convex, and if f'' exists on $]a, b[$ then we have

$$f''(x) \geq 0,$$

for any x such that $a < x < b$.

Definition 1.2. Let p and q be two real numbers belonging to $[1,+\infty]$. We say that p and q are conjugate exponents if

$$\frac{1}{p}+\frac{1}{q}=1.$$

This definition implies $1<p<\infty$ and $1<q<\infty$. Since $p=1$ we have $q=+\infty$, we say that 1 and $(+\infty)$ are conjugate exponents.

Theorem 1.1. *(Young's Inequality) Let a and b two positive real numbers, then we have*

$$ab\leq\frac{1}{p}a^p+\frac{1}{q}b^q,$$

with $p,q\in]1,+\infty[$ and p,q are conjugate exponents.

Proof. The function $\ln x$ is concave on $]0,+\infty[$
i.e., $\ln\left(\lambda_1x_1+\lambda_2x_2\right)\geq\lambda_1\ln\left(x_1\right)+\lambda_2\ln\left(x_2\right),\ \lambda_1+\lambda_2=1.$
So, for $\lambda_1=\frac{1}{p}$, $\lambda_2=\frac{1}{q}$ we have $\lambda_1+\lambda_2=\frac{1}{p}+\frac{1}{q}=1.$
we put $x_1=a^p\geq0$, $x_2=b^q\geq0$, then we have

$$\begin{aligned}\ln\left(\frac{1}{p}a^p+\frac{1}{q}b^q\right) &\geq \frac{1}{p}\ln\left(a^p\right)+\frac{1}{q}\ln\left(b^q\right)\\ &\geq \ln\left(a^p\right)^{\frac{1}{p}}+\ln\left(b^q\right)^{\frac{1}{q}}\\ &\geq \ln\left(a^p\right)^{\frac{1}{p}}\left(b^q\right)^{\frac{1}{q}}\\ &= \ln\left(ab\right)\end{aligned}$$

i.e.,

$$ab\leq\frac{1}{p}a^p+\frac{1}{q}b^q.$$

□

Theorem 1.2. *(Hölder's inequality) Let $(X,\mathcal{M},\mu)$ be a measure space, with f and g being two functions $f,\ g:X\to\overline{\mathbb{R}}$, measurable. Then we have*

$$\int_X|f(x)g(x)|\,d\mu(x)\leq\left(\int_X|f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}}\left(\int_X|g(x)|^q\,d\mu(x)\right)^{\frac{1}{q}},$$

where p and q are two conjugate exponents.

Proof. If one of the two terms in the product on the right-hand side of the inequality is zero or infinite, the inequality is automatically satisfied, so we may assume this is not the case. We put

$$F(x)=\frac{|f(x)|}{\left(\int_X|f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}}}\qquad\textit{and}\qquad G(x)=\frac{|g(x)|}{\left(\int_X|g(x)|^q\,d\mu(x)\right)^{\frac{1}{q}}},$$

we then have

$$\int_X (F(x))^p \, d\mu(x) = \int_X (G(x))^q \, d\mu(x) = 1.$$

If $x \in X$ is such that

$$0 < F(x) < \infty \qquad and \qquad 0 < G(x) < \infty,$$

there exist two real numbers t and u such that

$$F(x) = e^{\frac{t}{p}} \qquad and \qquad G(x) = e^{\frac{u}{q}}.$$

Since the function e^x is convex, and p and q are conjugate exponents, we have

$$e^{\frac{t}{p}+\frac{u}{q}} \leq \frac{1}{p}e^t + \frac{1}{q}e^u.$$

It follows that, for every x in X,

$$F(x)G(x) \leq \frac{1}{p}(F(x))^p + \frac{1}{q}(G(x))^q.$$

By integrating with respect to the measure μ, we have

$$\begin{aligned} \int_X F(x)G(x)d\mu(x) &\leq \frac{1}{p}\int_X (F(x))^p \, d\mu(x) + \frac{1}{q}\int_X (G(x))^q \, d\mu(x) \\ &\leq \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

i.e.,

$$\int_X \frac{|f(x)|}{\left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}} \times \frac{|g(x)|}{\left(\int_X |g(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}} d\mu(x) \leq 1,$$

so we have

$$\frac{1}{\left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}} \times \frac{1}{\left(\int_X |g(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}} \int_X |f(x)g(x)| \, d\mu(x) \leq 1.$$

i.e.,

$$\int_X |f(x)g(x)| \, d\mu(x) \leq \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \left(\int_X |g(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}.$$

□

Remark 1.1. When $p = q = 2$, Hölder's inequality is in this case known as the Cauchy-Schwarz inequality.

Theorem 1.3. *(Minkowski Inequality) Let $(X, \mathcal{M}, \mu)$ be a measure space, with, f and g being two* *functions,* $f,\ g : X \to \overline{\mathbb{R}}$*, measurable. We have* $\forall\ 1 \leq p \leq \infty$

$$\left(\int_X |f(x)+g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \leq \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_X |g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

Proof. We have

$$(f(x)+g(x))^p = f(x)\,(f(x)+g(x))^{p-1} + g(x)\,(f(x)+g(x))^{p-1}$$

i.e.,

$$|f(x)+g(x)|^p \leq |f(x)|\,|f(x)+g(x)|^{p-1} + |g(x)|\,|f(x)+g(x)|^{p-1}. \tag{1.1}$$

By applying Hölder's inequality, we have

$$\int_X |f(x)|\,|f(x)+g(x)|^{p-1} \, d\mu(x) \leq \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \left(\int_X |f(x)+g(x)|^{(p-1)q}\right)^{\frac{1}{q}},$$

and

$$\int_X |g(x)|\,|f(x)+g(x)|^{p-1} \, d\mu(x) \leq \left(\int_X |g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \left(\int_X |f(x)+g(x)|^{(p-1)q}\right)^{\frac{1}{q}}.$$

So, the equation (1.1) becomes

$$\int_X |f(x)+g(x)|^p \, d\mu(x) \leq \left(\left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_X |g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}\right) \left(\int_X |f(x)+g(x)|^{(p-1)q} \, d\mu(x)\right)^{\frac{1}{q}},$$

And since $(p-1)q = p$, we then have

$$\int_X |f(x)+g(x)|^p \, d\mu(x) \leq \left(\left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_X |g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}\right) \left(\int_X |f(x)+g(x)|^p \, d\mu(x)\right)^{\frac{1}{q}}.$$

We divide the two members by $\left(\int_X |f(x)+g(x)|^p \, d\mu(x)\right)^{\frac{1}{q}}$, we obtain the desired result. □

1.2 Elementary definitions and properties on $\mathcal{L}^p$ and L^p

1.2.1 the space $\mathcal{L}^p(X, \mathcal{M}, \mu)$

Definition 1.3. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we put

$$\mathcal{L}^p(X) = \left\{ f : (X, \mathcal{M}, \mu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda),\ f \text{ is measurable and } \int_X |f(x)|^p \, d\mu(x) < +\infty \right\}.$$

We note

$$\|f\|_{\mathcal{L}^p} = \|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}},$$

we will verify that $\|f\|_p$ is a semi-norm.

Particular case
If $(X, \mathcal{M}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), card)$, we note $\ell^p = \{x = (x_n)_{n\geq 0} : \sum_{n=0}^{\infty} |x_n|^p < \infty\}$, with the norm $\|x\|_{\ell^p} = \|x\|_p = (\sum_{n=0}^{\infty} |x_n|^p)^{\frac{1}{p}}$.

Remark 1.2.

$$\begin{aligned} f \in \mathcal{L}^p &\Leftrightarrow \|f\|_p < +\infty \; and \; measurable, \; because \\ f \in \mathcal{L}^p &\Leftrightarrow \int_X |f(x)|^p \, d\mu(x) < +\infty \; and \; measurable \\ &\Leftrightarrow \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} < +\infty \; and \; measurable \\ &\Leftrightarrow \|f\|_p < +\infty \; and \; measurable. \end{aligned}$$

Definition 1.4. A measurable function $f : X \to \mathbb{R}$ is said to be essentially bounded if there exists $M \geq 0$ such that $|f(x)| \leq M \; a.\, e.$

$$i.e., \; \mu\left(\{x \in X : |f(x)| > M\}\right) = 0.$$

We note by

$$\mathcal{L}^{\infty}(X) = \{f : X \to \mathbb{R}, \; f \; measurable \; such \; that \; \exists M \geq 0 : |f(x)| \leq M \; a.\, e. \; on \; X\}.$$

We also note, $\|f\|_{\mathcal{L}^{\infty}} = \|f\|_{\infty} = sup_{x\in X} ess \, |f(x)| = inf \, \{M : |f(x)| \leq M \; a.\, e. \; on \; X\}$.

Lemma 1.1. *Let $f \in \mathcal{L}^{\infty}(X, \mathcal{M}, \mu)$, so*

$$|f(x)| \leq \|f\|_{\infty} \; a.\, e. \; on \; X,$$

so that $\|f\|_{\infty} = inf \, \{M : |f(x)| \leq M \; a.\, e.\}$. In other words, $\|f\|_{\infty}$ is an attained bound.

Remark 1.3. With the above notation, $\|f\|_{\mathcal{L}^p} = \|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$ Hölder's and Minkowski's inequalities can be written as follows

$$\|f \cdot g\|_1 \leq \|f\|_p \, \|g\|_q, \quad \text{Hölder's inequality.}$$

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p, \quad \text{Minkowski's inequality.}$$

Corollary 1.1. *Let $p \in [1, +\infty]$, then $\mathcal{L}^p(X, \mathcal{M}, \mu)$ is a vector space over $\mathbb{R}$ and the mapping*

$$\begin{array}{rll} \|\cdot\|_p : & \mathcal{L}^p & \to \mathbb{R}_+ \\ & f & \longmapsto \|f\|_p \end{array}$$

is a semi-norm.

Proof. 1) $\mathcal{L}^p$ is a vector space over $\mathbb{R}$
a) $\forall f \in \mathcal{L}^p, \forall g \in \mathcal{L}^p \Rightarrow (f+g) \in \mathcal{L}^p$
Let $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^p$. Then, according to Minkowski's inequality, we have

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \tag{1.2}$$

Since $f \in \mathcal{L}^p \Leftrightarrow \|f\|_p < \infty$ and $g \in \mathcal{L}^p \Leftrightarrow \|g\|_p < \infty$.
So, (1.2) $\Leftrightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p < +\infty$.
i.e., $(f+g) \in \mathcal{L}^p$.
b) $\forall \lambda \in \mathbb{R}, \forall f \in \mathcal{L}^p \Rightarrow (\lambda f) \in \mathcal{L}^p$. We have

$$\begin{aligned} \|\lambda f\|_p &= \left(\int_X |\lambda f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \\ &= |\lambda| \, \|f\|_p \\ &< \infty. \end{aligned}$$

So, $(\lambda f) \in \mathcal{L}^p$.
i.e., $\mathcal{L}^p$ is a vector space on $\mathbb{R}$.
2) $\|f\|_p$ is a semi-norm, because
a) $\|f\|_p \geq 0$
b) $\|\lambda f\|_p = |\lambda| \, \|f\|_p$
c) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, Minkowski inequality.
d) We have

$$\begin{aligned} f = 0 \quad \Rightarrow \quad \|f\|_p &= \left(\int_X |0|^p \, d\mu(x) \right)^{\frac{1}{p}} \\ &= 0 \end{aligned}$$

and we also have

$$\begin{aligned} \|f\|_p = 0 \quad &\Rightarrow \quad \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} = 0 \\ &\Rightarrow \quad f = 0 \; a.\, e. \; on \; X. \end{aligned}$$

Hence the application $f \mapsto \|f\|_p$ is a semi-norm.

□

This result suggests decomposing the space $\mathcal{L}^p$ into equivalence classes in the following way: we say that two functions f and g belonging to $\mathcal{L}^p$ are equivalent if $\|f-g\|_p = 0$,
i.e., f and g are almost everywhere equal.

1.2.2 the space $L^p(X, \mathcal{M}, \mu)$

Let us consider the equivalence relation on $\mathcal{L}^p(X, \mathcal{M}, \mu)$ defined by

$$f\mathcal{R}g \Leftrightarrow f = g \; a.\, e. \quad i.e., \; \|f-g\|_p = 0.$$

We denote by $[f]$ the equivalence class of f for this notation

$$\begin{aligned}[f] &= \{g \in \mathcal{L}^p : g\mathcal{R}f\} \\ &= \{g \in \mathcal{L}^p : f = g \; a.\, e.\}\end{aligned}$$

Definition 1.5. Let $p \in [1, +\infty]$. We note $L^p(X, \mathcal{M}, \mu)$ the quotient $\mathcal{L}^p(X, \mathcal{M}, \mu)$ by the equivalence relation $\mathcal{R}$

$$\begin{aligned}L^p &= \{[f], \; f \in \mathcal{L}^p\} \\ &= \mathcal{L}^p/\mathcal{R}.\end{aligned}$$

If we put $\|[f]\|_p = \|f\|_p$, we obtain

$$\begin{aligned}\|[f]\|_p = 0 &\Leftrightarrow \|f\|_p = 0 \\ &\Leftrightarrow f = 0 \; a.\, e. \\ &\Leftrightarrow [f] = [0].\end{aligned}$$

We associate $L^p(X)$ with the following two operations

$$+ : L^p(X) \times L^p(X) \to L^p(X) \quad and \quad \bullet : \mathbb{R} \times L^p(X) \to L^p(X)$$

defined by: $[f] + [g] = [f+g] \quad and \quad \forall \lambda \in \mathbb{R}; \lambda [f] = [\lambda f]$,
We obtain a new vector space $(L^p(X), +, \bullet)$ over the field $\mathbb{R}$.

Remark 1.4. 1) We consider the elements of $L^p(X)$ (the set of equivalence classes) as ordinary functions, and we write f instead of $[f]$.

2) The mapping

$$\begin{aligned} \|\cdot\|_p : L^p(X) &\rightarrow \mathbb{R}_+ \\ f &\mapsto \|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \end{aligned}$$

is a norm on $L^p(X)$.

Theorem 1.4. *(Riesz-Fisher Theorem) The Lebesgue space $L^p(X, \mathcal{M}, \mu)$ is a Banach space (a complete normed vector space) for every $p \in [1, +\infty]$, with the norm*

$$1 \leq p < \infty, \; we \; have \; \|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}, \; p = +\infty \; we \; have \; \|f\|_\infty = sup_{x \in X} ess \, |f(x)| \, .$$

Proof. See [3]. □

Convergence in $L^p((X, \mathcal{M}, \mu)$ Let $(f_n)_{n \geq 0}$ be a sequence of functions in L^p, and let $f \in L^p$. We say that the sequence $(f_n)_{n \geq 0}$ converges to f in L^p and we write $f_n \overset{L^p}{\rightarrow} f$, if

$$lim_{n \rightarrow +\infty} \|f_n - f\|_p = 0.$$

Exercise 1.1. Let $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p([0, +\infty[)$ and $g \in L^q([0, +\infty[)$, calculate

$$lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(x) g(x) dx$$

Solution We have

$$\begin{aligned} \left| \int_0^T f(x) g(x) dx \right| &\leq \int_0^T |f(x) g(x)| \, dx \\ &\leq \int_0^{+\infty} |f(x) g(x)| \, dx. \end{aligned}$$

According to Hölder's inequality, we have

$$\begin{aligned} \left| \int_0^T f(x) g(x) dx \right| &\leq \left(\int_0^{+\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_0^{+\infty} |g(x)|^q \, dx \right)^{\frac{1}{q}} \\ &\leq \|f\|_p \, \|g\|_q \, . \end{aligned}$$

So we have

$$\frac{1}{T} \left| \int_0^T f(x) g(x) dx \right| \leq \frac{1}{T} \|f\|_p \, \|g\|_q \, .$$

i.e.,

$$lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(x) g(x) dx = 0.$$

Corollary 1.2. *The space $L^2(X, \mathcal{M}, \mu)$ is a Hilbert space, equipped with the scalar product*

$$< f, g >= \int_X f(x)g(x)d\mu(x),$$

where $f, g : X \to \mathbb{R}$ and $f, g \in L^2(X, \mathcal{M}, \mu)$.

Cauchy-Schwarz inequality:
Hölder's inequality in the case $p = 2$, gives the Cauchy-Schwartz inequality. We have

$$\|f \cdot g\|_1 \leq \|f\|_2 \|g\|_2 ,$$

so we have

$$\int_X |f(x)g(x)| \, d\mu(x) \leq \left(\int_X |f(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}} \left(\int_X |g(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}},$$

i.e.,

$$|< f, g >| \leq \|f\|_2 \cdot \|g\|_2 .$$

Remark 1.5. For $p \neq 2$, the space $L^p(X, \mathcal{M}, \mu)$ is not a Hilbert space.

Theorem 1.5. *1) Let $(X, \mathcal{M}, \mu)$ be a finite measure space (i.e., $\mu(X) < +\infty$) and let $p, q \in [1, +\infty]$ with $1 \leq q \leq p$. Then we have*

$$L^p(X) \subset L^q(X).$$

2) If $1 \leq p_1 < p_2$, we have

$$\ell^1 \subset \ell^{p_1} \subset \ell^{p_2}.$$

Proof. 1) Assume that $1 \leq q < p$ (since the case $q = p$ is trivial).
We put $r = \frac{p}{q} > 1$ and r' such that $\frac{1}{r} + \frac{1}{r'} = 1$.
Let $f \in L^p$, so we have

$$\begin{aligned} \int_X |f|^{qr} \, d\mu &= \int_X |f|^p \, d\mu \\ &< +\infty, \end{aligned}$$

i.e., $(f)^q \in L^r$.
And

$$\begin{aligned} \int_X |1|^{r'} \, d\mu &= \mu(X) \\ &< +\infty, \end{aligned}$$

i.e., $1 \in L^{r'}$.

Hölder's inequality applied to $(f)^q$ and 1, we obtain

$$\begin{aligned}\int_X |f|^q \times 1 d\mu &\leq \left(\int_X |f|^{qr}\, d\mu\right)^{\frac{1}{r}} \left(\int_X |1|^{r'}\, d\mu\right)^{\frac{1}{r'}} \\ &\leq (\int_X |f|^p\, d\mu)^{\frac{1}{r}} (\mu(X))^{\frac{1}{r'}}.\end{aligned}$$

Which implies that

$$\|f\|_q^q \leq \|f\|_p^q \,(\mu(X))^{1-\frac{q}{p}},$$

so we have

$$\|f\|_q \leq \|f\|_p \,(\mu(X))^{\frac{1}{q}-\frac{1}{p}},$$

i.e.,

$$L^p(X) \subset L^q(X).$$

2) We show that if, $p_1 < p_2$ we have $\ell^{p_1} \subset \ell^{p_2}$

Let $x = (x_n)_{n\geq 0} \in \ell^{p_1} \Leftrightarrow \sum_{n=0}^{\infty} |x_n|^{p_1} < \infty$

i.e., the series is absolutely convergent $\Rightarrow$ the series is convergent.

So according to the necessary condition for the convergence of a series, we have

$$lim_{n\to+\infty} x_n = 0 \Leftrightarrow \forall \epsilon > 0; \exists n_0 \in \mathbb{N} : \forall n \geq n_0; |x_n| < \epsilon = 1.$$

We have

$$\begin{aligned}p_1 < p_2 &\Rightarrow |x_n|^{p_2} < |x_n|^{p_1}\,,\ \forall n \geq n_0 \\ &\Rightarrow \sum_{n=n_0}^{\infty} |x_n|^{p_2} < \sum_{n=n_0}^{\infty} |x_n|^{p_1} \\ &\Rightarrow \sum_{n=0}^{\infty} |x_n|^{p_2} < \infty,\end{aligned}$$

i.e., $x = (x_n)_{n\geq 0} \in \ell^{p_2}$.

For the first inclusion, it is enough to replace the pair (p_1, p_2) with $(1, p_1)$. □

Example 1.1. Let

$$\begin{aligned} f : (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) &\longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \\ x &\mapsto f(x) = \frac{1}{1+|x|},\ \lambda \text{ is the Lebesgue measure.}\end{aligned}$$

1) Show that $f \in L^2(\mathbb{R})$ and that $f \notin L^1(\mathbb{R})$.

2) What can we deduce?

Solution We have that f is continuous on $\mathbb{R} \Rightarrow f$ is measurable.
moreover,

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|^2 \, dx &= \int_{\mathbb{R}} \frac{1}{1+2|x|+x^2} dx \\ &\leq \int_{\mathbb{R}} \frac{1}{1+x^2} dx = \pi \\ &< \infty, \end{aligned}$$

So, $f \in L^2(\mathbb{R})$.
But,

$$\begin{aligned} \int_{\mathbb{R}} |f(x)| \, dx &= \int_{\mathbb{R}} \frac{1}{1+|x|} dx \\ &= 2\int_0^{+\infty} \frac{1}{1+x} dx = +\infty \end{aligned}$$

So, $f \notin L^1(\mathbb{R})$.
2) We deduce that $L^2(\mathbb{R}) \nsubseteq L^1(\mathbb{R})$, because $\lambda(\mathbb{R}) = +\infty$.

Theorem 1.6. ***(Extension of the Dominated Convergence Theorem (DCT) of Lebesgue)***
Let $p \in [1, +\infty[$, $(f_n)_{n\geq 0}$ a sequence of elements of L^p and $g : X \to \overline{\mathbb{R}}_+$ an element of L^p such that

(i) $lim_{n\to+\infty} f_n(x) = f(x)$ $a.$ $e.$ with respect to μ,

(ii) $|f_n(x)| \leq g(x)$ $a.$ $e.$ with respect to μ.

Then, $f \in L^p$ and $lim_{n\to+\infty} \|f_n - f\|_p = 0$

Corollary 1.3. *Let $(f_n)_{n\geq 0}$ be a sequence of elements in L^p such that*

$$\sum_{n=0}^{\infty} \|f_n\|_p < +\infty.$$

Then the series $\sum_{n=0}^{\infty} f_n(x)$ is absolutely convergent almost everywhere $(a.$ $e.)$ with respect to μ. Moreover, the function $f(x) = \sum_{n=0}^{\infty} f_n(x)$ defined almost everywhere with respect to μ belongs to L^p and $lim_{n\to+\infty} \|\sum_{k=0}^{n} f_k - f\|_p = 0$.

1.3 Density theorems

We will establish that certain sets of particularly simple functions are dense in the L^p spaces.

Definition 1.6. Let $(E, \|\cdot\|)$ be a normed space and E_0 a subspace of E, We say that E_0 is dense in E, if, for every $f \in E$ and every $\epsilon > 0$ there exists an element $f_0 \in E_0$ such that $\|f - f_0\| < \epsilon$.

Theorem 1.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space. The set E of step functions defined on this space, such that*

$$(\forall f \in E); \quad \mu(\{x \in X : f(x) \neq 0\}) < \infty,$$

is dense in $L^p(X)$, for $1 \leq p < \infty$.

Proof. The definition of the set E implies that $E \subset L^p$. Let f be a positive function in L^p and let $(e_n)_{n\geq 0}$ be an increasing sequence of positive step functions converging to f. For every n we have $0 \leq e_n < f$ which implies that $e_n \in L^p$, and consequently, $e_n \in E$. Furthermore, the inequality $|f - e_n|^p \leq f^p$ allows us to apply Lebesgue's Dominated Convergence Theorem. It follows that

$$lim_{n\to\infty} \|f - e_n\|_p = 0.$$

f belongs to the closure of E. Since any real (or complex) function can be written as a linear combination of two (or four) positive functions, we deduce that L^p coincides with the closure of E, or equivalently that, E is dense in L^p. □

• We now present some results concerning density in the space L^p, in the case where $X = \mathbb{R}$ and μ is the Lebesgue measure λ.

Definition 1.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The support of f is defined as the closure of the open set $\{x \in \mathbb{R} : f(x) \neq 0\}$, i.e.,

$$supp\ f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}.$$

Corollary 1.4. *The vector space $C_c^0(\mathbb{R})$ of continuous functions with compact support on $\mathbb{R}$ is dense in L^p for every $p \in [1, +\infty[$.*

Example 1.2. The function

$$\varphi(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & , |x| < 1 \\ 0 & , |x| \geq 1 \end{cases}$$

belongs to the space $C_c^0(\mathbb{R})$, because the function is continuous on $\mathbb{R}$, and its support is $supp\ \varphi = [-1, 1]$.

Corollary 1.5. *The vector space $C_c^k(\mathbb{R})$ of functions of class C^k with bounded support on $\mathbb{R}$ is dense in L^p for all $p \in [1, +\infty[$.*

Remarks 1.2. 1) k being arbitrary, note that in particular the space of indefinitely differentiable functions with bounded support on $\mathbb{R}$ (i.e., the space of test functions, denoted $\mathcal{D}(\mathbb{R})$) is dense

in L^p for all $p \in [1, +\infty[$.
2) The results we obtained are only valid for $1 \leq p < \infty$. Thus, for example the constant function $f(x) = 1$ which belongs to L^∞ does not belong to the closure of $C_c^0(\mathbb{R})$.

1.4 Some properties of the space $L^p(X)$

The table below presents some properties of the space L^p (Reflexivity, Separability, Dual of L^p).

The space $L^p(X)$	Reflexive	Separable	Dual space
L^p, $1 < p < \infty$	Yes	Yes	L^q, with $\frac{1}{p} + \frac{1}{q} = 1$
L^1	No	Yes	L^∞
L^∞	No	No	strictly contains L^1

Exercise 1.2. 1) Let $(X, \mathcal{M}, \mu)$ be a measured space, and let f and g be two functions belonging respectively to $L^p(X)$ and $L^q(X)$ where p and q are positive. Show that if we put $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then we have

$$f \cdot g \in L^r(X) \quad and \quad \|fg\|_r \leq \|f\|_p \|g\|_q .$$

2) Now let $p, q \in]1, +\infty[$ such that $pq \geq p + q$, suppose that $(f_n)_{n\geq 0} \in L^p$ and $(g_n)_{n\geq 0} \in L^q$ such that

$$f_n \xrightarrow{L^p} f \quad and \quad g_n \xrightarrow{L^q} g.$$

Find the appropriate space such that the sequence $(f_n g_n)_{n\geq 0}$ converges in that space?

Solution 1) We have

$$\|fg\|_r^r = \int_X |f|^r |g|^r \, d\mu \tag{1.3}$$

We have: $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \Rightarrow \frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$.
According to Hölder's inequality, we have

$$\begin{aligned}
(1.3) \Leftrightarrow \|fg\|_r^r &= \int_X |f|^r |g|^r \, d\mu \\
&\leq (\int_X (|f|^r)^{\frac{p}{r}} d\mu)^{\frac{r}{p}} (\int_X (|g|^r)^{\frac{q}{r}} d\mu)^{\frac{r}{q}} \\
&\leq \|f\|_p^r \|g\|_q^r ,
\end{aligned}$$

i.e.,

$$\|fg\|_r \leq \|f\|_p \|g\|_q .$$

We have $f \in L^p \Leftrightarrow f$ is measurable and $\|f\|_p < \infty$, $g \in L^q \Leftrightarrow g$ is measurable and $\|g\|_q < \infty$, so $(f \cdot g)$ is measurable and $\|fg\|_r < \infty$, which implies that $f \cdot g \in L^r(X)$.
2) We have

$$f_n g_n - fg = (f_n - f)(g_n - g) + (f_n - f)g + f(g_n - g).$$

According to Minkowski's inequality, we have

$$\|f_ng_n - fg\|_r \leq \|(f_n - f)(g_n - g)\|_r + \|(f_n - f)g\|_r + \|f(g_n - g)\|_r .$$

According to Hölder's inequality, we have

$$\|f_ng_n - fg\|_r \leq \|f_n - f\|_p \|g_n - g\|_q + \|f_n - f\|_p \|g\|_q + \|f\|_p \|g_n - g\|_q .$$

Then we have

$$lim_{n\to+\infty} \|f_ng_n - fg\|_r = 0,$$

which implies that $f_ng_n \overset{L^r}{\to} fg$ for $r = \frac{pq}{p+q}$.

Chapter 2

Fourier Transform

In analysis, the Fourier transform is an extension, for non-periodic functions, of the Fourier series expansion of periodic functions. The Fourier transform associates, to an integrable function defined on $\mathbb{R}$ and taking real or complex values, another function on $\mathbb{R}$ called the Fourier transform, whose independent variable can be interpreted in physics as frequency or angular frequency.

The Fourier transform represents a function by the spectral density from which it originates, as an average of trigonometric functions of all frequencies. Measure theory as well as distribution theory provide rigorous foundations for the definition of the Fourier transform in its full generality; it plays a fundamental role in harmonic analysis. When a function represents a physical phenomenon, such as the state of an electromagnetic field or an acoustic field at a point, it is called a signal, and its Fourier transform is called its spectrum.

In this chapter, we will study the Fourier transform of summable and square-integrable functions, along with some properties and applications for solving integral equations and partial differential equations.

2.1 Definitions and Notations

We denote by $L^1(\mathbb{R})$, the set of measurable functions defined from $\mathbb{R}$ to $\mathbb{R}$, such that

$$\int_{\mathbb{R}} |f(x)|\, dx < +\infty$$

.

Examples 2.1. 1) The function $f(x) = \frac{1}{1+x^2}$ belongs to $L^1(\mathbb{R})$, because f is measurable and

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|\, dx &= \int_{\mathbb{R}} \left|\frac{1}{1+x^2}\right| dx \\ &= \int_{\mathbb{R}} \frac{1}{1+x^2} dx = \pi < +\infty\end{aligned}$$

2) The function g from $\mathbb{R}$ to $\mathbb{R}$ defined by $g(x) = x$ does not belong to $L^1(\mathbb{R})$. In general, except in the case of the zero function, polynomial functions do not belong to $L^1(\mathbb{R})$.

Definition 2.1. Let $f \in L^1(\mathbb{R})$. The Fourier transform of f is the complex-valued function of the real variable ξ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}.$$

This integral is well-defined because $\left|e^{-2\pi i x\cdot\xi} f(x)\right| = |f(x)|$ and $f \in L^1(\mathbb{R})$.
We will symbolically write, $\widehat{f}(\xi) = \mathcal{F}(f(x))(\xi)$.

Proposition 2.1. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is bounded and continuous, $\widehat{f}(\xi)$ tends to 0 as $|\xi| \to +\infty$, and* $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$.

Proof. We have

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}.$$

The function under the integral sign is continuous for almost every $x \in \mathbb{R}$ and measurable for every $\xi \in \mathbb{R}$. Moreover, we have

$$\left|e^{-2\pi i x\cdot\xi} f(x)\right| = |f(x)|\,,\ \forall \xi \in \mathbb{R}.$$

The second term belongs to $L^1(\mathbb{R})$, and by the continuity theorem for functions defined by integrals, the function $\widehat{f}$ is continuous. Moreover, we have

$$\begin{aligned}\left|\widehat{f}(\xi)\right| &\leq \int_{\mathbb{R}} |f(x)|\, dx \\ &= \|f\|_1\,,\ \forall \xi \in \mathbb{R}\end{aligned}$$

i.e.,

$$\begin{aligned}\left\|\widehat{f}\right\|_\infty &= sup_{\xi\in\mathbb{R}} ess \left|\widehat{f}(\xi)\right| \\ &\leq \|f\|_1 \\ &< +\infty.\end{aligned}$$

This shows that $\widehat{f}$ is bounded.

- We now show that $lim_{|\xi|\to+\infty} \widehat{f}(\xi) = 0$.

It is known that the space of functions with bounded support on $\mathbb{R}$ is dense in $L^1(\mathbb{R})$, i.e., $\forall f \in L^1(\mathbb{R}),\ \exists(\varphi_n)_{n\geq 0}$ a sequence of functions with bounded support such that

$$\varphi_n \overset{L^1}{\to} f \Leftrightarrow lim_{n\to+\infty} \|\varphi_n - f\|_1 = 0.$$

Then we have, $\varphi(x) = \sum_{k=1}^{n} \alpha_k \chi_{[\alpha_{k-1},\alpha_k]}(x)$,
which implies that

$$\begin{aligned}
\widehat{\varphi}(\xi) &= \int_{\mathbb{R}} (\sum_{k=1}^{n} \alpha_k \chi_{[\alpha_{k-1},\alpha_k]}(x)) e^{-2\pi i x\cdot\xi} dx \\
&= \sum_{k=1}^{n} \alpha_k \int_{\alpha_{k-1}}^{\alpha_k} e^{-2\pi i x\cdot\xi} dx \\
&= \sum_{k=1}^{n} \frac{i\alpha_k}{2\pi\xi} (e^{-2\pi i\xi\alpha_k} - e^{-2\pi i\xi\alpha_{k-1}}),
\end{aligned}$$

so we have

$$|\widehat{\varphi}(\xi)| \leq \frac{1}{\pi}(\sum_{k=1}^{n} |\alpha_k|)\frac{1}{|\xi|}.$$

If $|\xi| \to \infty$, on a $|\widehat{\varphi}(\xi)| \to 0$, so

$$\left|\widehat{f}(\xi)\right| \leq \left|\widehat{f}(\xi) - \widehat{\varphi}(\xi)\right| + |\widehat{\varphi}(\xi)|\,.$$

Which implies that $lim_{|\xi|\to+\infty} \left|\widehat{f}(\xi)\right| = 0$,
i.e.,

$$lim_{|\xi|\to+\infty} \widehat{f}(\xi) = 0.$$

□

Example 2.1. Calculate the Fourier transform of the rectangular function:

$$\pi(x) = \begin{cases} 1, & |x| \leq \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$$

.

Solution We have

$$\begin{aligned}\widehat{\pi}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi}\cdot\pi(x)dx\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x\cdot\xi}dx\\ &= \frac{-1}{2\pi i\xi}(e^{-\pi i\xi}-e^{\pi i\xi})\\ &= \frac{1}{\pi\xi}(\frac{e^{\pi i\xi}-e^{-\pi i\xi}}{2i}),\end{aligned}$$

so,

$$\widehat{\pi}(\xi)=\frac{\sin(\pi\xi)}{\pi\xi},\quad \text{car } \sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}, \text{ avec } \theta=\pi\xi.$$

2.1.1 Particular case 1: if f is an even function

We know that $e^{i\theta}=\cos(\theta)+i\sin(\theta)$, so the Fourier integral can be written as

$$\mathcal{F}(f(x))(\xi)=\int_{-\infty}^{+\infty}(\cos(2\pi x\xi)-i\sin(2\pi x\xi))f(x)dx.$$

Now, the functions $x\mapsto f(x)\cos(2\pi x\xi)$ and $x\mapsto f(x)\sin(2\pi x\xi)$ are even and odd functions, respectively. Therefore,

$$\int_{\mathbb{R}} f(x)\cos(2\pi x\xi)dx=2\int_0^{+\infty} f(x)\cos(2\pi x\xi)dx \quad \text{and} \quad \int_{\mathbb{R}} f(x)\sin(2\pi x\xi)dx=0.$$

Hence, if f is even, $\mathcal{F}(f(x))(\xi)$ is a real number, and

$$\mathcal{F}(f(x))(\xi)=2\int_0^{+\infty} f(x)\cos(2\pi x\xi)dx.$$

2.1.2 Particular case 2: if f is an odd function

In the same way, we can show that if f is odd, then $\mathcal{F}(f(x))(\xi)$ is a purely imaginary number, and we have

$$\mathcal{F}(f(x))(\xi)=-2i\int_0^{+\infty} f(x)\sin(2\pi x\xi)dx.$$

2.2 Inverse Fourier Transform

We can obtain $f(x)$ from $\widehat{f}(\xi)$ using the inverse transform (also called the Fourier inversion formula):

$$\begin{aligned} f(x) &= \mathcal{F}^{-1}(\widehat{f}(\xi))(x) \\ &= \int_{\mathbb{R}} e^{2\pi i x\cdot\xi}\widehat{f}(\xi)d\xi. \end{aligned}$$

More generally, if f is not continuous at x_0, we have

$$\int_{-\infty}^{+\infty} e^{2\pi i x\cdot\xi}\widehat{f}(\xi)d\xi = \frac{f(x_0+0)+f(x_0-0)}{2},$$

where $f(x_0+0)$ and $f(x_0-0)$ are the right-hand and left-hand limits of $f(x)$.

Exercise 2.1. 1) Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$
2) Using the inverse Fourier transform, calculate the integral $\int_{-\infty}^{+\infty} \frac{\cos(2\pi x\xi)\sin(2\pi a\xi)}{\xi}d\xi$.
3) Deduce the value of the integral $\int_0^{+\infty} \frac{\sin(x)}{x}dx$.

Solution We have

$$\begin{aligned} \widehat{f}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi}\cdot f(x)dx \\ &= \int_{-a}^{a} e^{-2\pi i x\cdot\xi}dx \\ &= \frac{-1}{2\pi i\xi}(e^{-2\pi i a\xi} - e^{2\pi i a\xi}) \\ &= \frac{1}{\pi\xi}(\frac{e^{2\pi i a\xi} - e^{-2\pi i a\xi}}{2i}) \end{aligned}$$

so,

$$\widehat{f}(\xi) = \frac{\sin(2\pi a\xi)}{\pi\xi}, \quad \text{because } \sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}, \text{ with } \theta = 2\pi a\xi.$$

2) According to the Fourier inversion formula, we have

$$\int_{\mathbb{R}} e^{2\pi i x\cdot\xi}\widehat{f}(\xi)d\xi = f(x),$$

which implies that

$$\int_{\mathbb{R}} (\cos(2\pi x\xi) + i\sin(2\pi x\xi))\widehat{f}(\xi)d\xi = \begin{cases} 1, & |x| < a \\ \frac{1}{2}, & |x| = a \\ 0, & |x| > a \end{cases}$$

so we have

$$\int_{\mathbb{R}} \frac{\cos(2\pi x\xi)\sin(2\pi a\xi)}{\pi\xi}d\xi + i\int_{\mathbb{R}} \frac{\sin(2\pi x\xi)\sin(2\pi a\xi)}{\pi\xi}d\xi = \begin{cases} 1, & |x| < a \\ \frac{1}{2}, & |x| = a \\ 0, & |x| > a \end{cases} \tag{2.1}$$

Now, the function $\xi \mapsto \frac{\sin(2\pi x\xi)\sin(2\pi a\xi)}{\pi\xi}$ is an odd function, so the relation (2.1) becomes

$$\int_{\mathbb{R}} \frac{\cos(2\pi x\xi)\sin(2\pi a\xi)}{\xi}d\xi = \begin{cases} \pi, & |x| < a \\ \frac{\pi}{2}, & |x| = a \\ 0, & |x| > a \end{cases}$$

3) For $x = 0$ and $a = \frac{1}{2\pi}$, we have

$$\int_{\mathbb{R}} \frac{\sin(\xi)}{\xi}d\xi = \pi \Rightarrow 2\int_0^{+\infty} \frac{\sin(\xi)}{\xi}d\xi = \pi$$

i.e.,

$$\int_0^{+\infty} \frac{\sin(\xi)}{\xi}d\xi = \frac{\pi}{2}.$$

2.3 Properties of the Fourier transform

1) **Linearity** Let $f,\ g \in L^1(\mathbb{R})$ and $\alpha,\ \beta \in \mathbb{R} \vee \mathbb{C}$, then

$$\mathcal{F}(\alpha f(x) + \beta g(x))(\xi) = \alpha\widehat{f}(\xi) + \beta\widehat{g}(\xi).$$

Proof. We have

$$\begin{aligned} \mathcal{F}(\alpha f(x) + \beta g(x))(\xi) &= \int_{\mathbb{R}} e^{-2\pi ix\cdot\xi}(\alpha f(x) + \beta g(x))dx \\ &= \alpha\int_{\mathbb{R}} e^{-2\pi ix\cdot\xi}f(x)dx + \beta\int_{\mathbb{R}} e^{-2\pi ix\cdot\xi}g(x)dx \\ &= \alpha\widehat{f}(\xi) + \beta\widehat{g}(\xi) \end{aligned}$$

□

2) **Translation** Let $f \in L^1(\mathbb{R})$ and $a \in \mathbb{R}$, then

$$\mathcal{F}(\tau_a f)(\xi) = e^{-2\pi i a\xi}\widehat{f}(\xi),$$

where $(\tau_a f)(x) = f(x-a)$.

Proof. We have

$$\begin{aligned}\mathcal{F}(f(x-a))(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x-a)dx, \text{ we put } y = x-a \\ &= \int_{\mathbb{R}} e^{-2\pi i (y+a)\xi} f(y)dy \\ &= e^{-2\pi i a\xi}\int_{\mathbb{R}} e^{-2\pi i y\xi} f(y)dy \\ &= e^{-2\pi i a\xi}\widehat{f}(\xi)\end{aligned}$$

□

3) **Change of scale** Let $f \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{R}^\star$, then

$$\mathcal{F}(h_\lambda f)(\xi) = \frac{1}{|\lambda|}\widehat{f}(\frac{\xi}{\lambda}),$$

where $(h_\lambda f)(x) = f(\lambda x)$.

Proof. We have

$$\mathcal{F}(h_\lambda f)(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(\lambda x)dx, \text{ There are two cases}$$

a) $\lambda > 0$, we put $\lambda x = y \Leftrightarrow dy = \lambda dx$, so we have

$$\begin{aligned}\mathcal{F}(f(\lambda x))(\xi) &= \int_{\mathbb{R}} e^{-2\pi i y\frac{\xi}{\lambda}} f(y)\frac{dy}{\lambda} \\ &= \frac{1}{\lambda}\int_{\mathbb{R}} e^{-2\pi i y(\frac{\xi}{\lambda})} f(y)dy \\ &= \frac{1}{\lambda}\widehat{f}(\frac{\xi}{\lambda})\end{aligned}$$

b) $\lambda < 0$, so we have $\lambda x = y \Leftrightarrow dy = \lambda dx$

$$\begin{aligned}\mathcal{F}(f(\lambda x))(\xi) &= \int_{+\infty}^{-\infty} e^{-2\pi i y \frac{\xi}{\lambda}} f(y) \frac{dy}{\lambda} \\ &= \frac{-1}{\lambda} \int_{-\infty}^{+\infty} e^{-2\pi i y (\frac{\xi}{\lambda})} f(y) dy \\ &= \frac{-1}{\lambda} \widehat{f}(\frac{\xi}{\lambda}) \\ &= \frac{1}{-\lambda} \widehat{f}(\frac{\xi}{\lambda}),\end{aligned}$$

so we have

$$\mathcal{F}(f(\lambda x))(\xi) = \frac{1}{|\lambda|} \widehat{f}(\frac{\xi}{\lambda}).$$

□

4) **Modulation** Let $f \in L^1(\mathbb{R})$ and $\xi_0 \in \mathbb{R}$, then

$$\mathcal{F}(e^{2\pi i \xi_0 x} f(x))(\xi) = \widehat{f}(\xi - \xi_0).$$

Proof. We have

$$\begin{aligned}\mathcal{F}(e^{2\pi i \xi_0 x} f(x))(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x \cdot \xi} \cdot e^{2\pi i \xi_0 x} f(x) dx \\ &= \int_{\mathbb{R}} e^{-2\pi i (\xi - \xi_0) x} f(x) dx \\ &= \widehat{f}(\xi - \xi_0).\end{aligned}$$

□

Remark 2.1. If $f \in L^1(\mathbb{R})$, then $lim_{x \to +\infty} f(x) = lim_{x \to -\infty} f(x) = 0$.

Proposition 2.2. *Let $f \in L^1(\mathbb{R})$, and suppose that f is differentiable and $f' \in L^1(\mathbb{R})$, then*

$$\mathcal{F}(f'(x))(\xi) = (2\pi i \xi) \widehat{f}(\xi).$$

Moreover, if f has derivatives up to order n that all belong to $L^1(\mathbb{R})$, then

$$\mathcal{F}(f^{(n)}(x))(\xi) = (2\pi i \xi)^n \widehat{f}(\xi).$$

Proof. We have

$$\begin{aligned}\mathcal{F}(f'(x))(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f'(x)dx \\ &= [e^{-2\pi i x\cdot\xi} f(x)]_{-\infty}^{+\infty} + (2\pi i\xi)\int_{-\infty}^{+\infty} e^{-2\pi i x\cdot\xi} f(x)dx, \text{ by parts} \\ &= 0 + (2\pi i\xi)\widehat{f}(\xi), \text{ because } e^{-2\pi i x\cdot\xi} f(x) \in L^1(\mathbb{R}).\end{aligned}$$

□

• We can show by recurrence that

$$\mathcal{F}(f^{(n)}(x))(\xi) = (2\pi i\xi)^n \widehat{f}(\xi).$$

Proposition 2.3. *Let $f \in L^1(\mathbb{R})$. If $xf(x) \in L^1(\mathbb{R})$, then $\widehat{f}$ is differentiable and we have*

$$\frac{d}{d\xi}\widehat{f}(\xi) = \mathcal{F}(-2\pi i x f(x))(\xi),$$

if further, $x^n f(x) \in L^1(\mathbb{R})$ then

$$\frac{d^{(n)}}{d\xi^n}\widehat{f}(\xi) = \mathcal{F}((-2\pi i x)^n f(x))(\xi).$$

Proof. We have

$$\frac{d}{d\xi}\widehat{f}(\xi) = \frac{d}{d\xi}\int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx,$$

Since $\left|-2\pi i x f(x) e^{-2\pi i x\cdot\xi}\right| = 2\pi\,|xf(x)|$ and by assumption $xf(x) \in L^1(\mathbb{R})$, then according to the derivation theorem under the integral sign,

$$\begin{aligned}\frac{d}{d\xi}\widehat{f}(\xi) &= \int_{\mathbb{R}} \frac{d}{d\xi} e^{-2\pi i x\cdot\xi} f(x)dx \\ &= \int_{\mathbb{R}} -2\pi i x e^{-2\pi i x\cdot\xi} f(x)dx \\ &= \mathcal{F}(-2\pi i x f(x))(\xi).\end{aligned}$$

□

• More Generally, if $x^n f(x) \in L^1(\mathbb{R})$ we can show by recurrence that

$$\frac{d^{(n)}}{d\xi^n}\widehat{f}(\xi) = \mathcal{F}((-2\pi i x)^n f(x))(\xi).$$

2.4 Convolution product

Let $f,\ g \in L^1(\mathbb{R})$. Then the convolution $(f \star g) \in L^1(\mathbb{R})$ and we have

$$(f \star g)(x) = \int_{\mathbb{R}} f(t)g(x-t)dt.$$

The Fourier transform of the convolution product is

$$\mathcal{F}((f \star g)(x))(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi).$$

We can also show that

$$\mathcal{F}^{-1}((f \star g)(x))(\xi) = \mathcal{F}^{-1}(f(x))(\xi) \cdot \mathcal{F}^{-1}(g(x))(\xi).$$

Proof. We have

$$\begin{aligned}\mathcal{F}((f \star g)(x))(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi}(f \star g)(x)dx \\ &= \int_{-\infty}^{+\infty} e^{-2\pi i x\cdot\xi}(\int_{-\infty}^{+\infty} f(t)g(x-t)dt)dx.\end{aligned}$$

By applying Fubini's Theorem, we obtain

$$\int_{-\infty}^{+\infty} e^{-2\pi i x\cdot\xi}(\int_{-\infty}^{+\infty} f(t)g(x-t)dt)dx = \int_{-\infty}^{+\infty} f(t)(\int_{-\infty}^{+\infty} e^{-2\pi i x\cdot\xi} g(x-t)dx)dt.$$

We put $y = x - t \Leftrightarrow dy = dx$, then

$$\begin{aligned}\mathcal{F}((f \star g)(x))(\xi) &= \int_{-\infty}^{+\infty} f(t)(\int_{-\infty}^{+\infty} e^{-2\pi i(y+t)\xi} g(y)dy)dt \\ &= \int_{-\infty}^{+\infty} e^{-2\pi i t\cdot\xi} f(t)(\int_{-\infty}^{+\infty} e^{-2\pi i y\cdot\xi} g(y)dy)dt \\ &= \int_{-\infty}^{+\infty} e^{-2\pi i t\cdot\xi} f(t)dt \int_{-\infty}^{+\infty} e^{-2\pi i y\cdot\xi} g(y)dy \\ &= \widehat{f}(\xi) \cdot \widehat{g}(\xi).\end{aligned}$$

□

2.5 Parseval-Plancherel formula

We have the following relation established by Parseval for Fourier series and generalized by Plancherel (1910) to Fourier transforms

$$\int_{-\infty}^{+\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{+\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi.$$

An important special case if $f = g$, we have

$$\int_{-\infty}^{+\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{+\infty} \widehat{f}(\xi)\overline{\widehat{f}(\xi)}d\xi,$$

i.e.,

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_{-\infty}^{+\infty} \left|\widehat{f}(\xi)\right|^2 d\xi.$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}} f(x)\overline{g(x)}dx &= \mathcal{F}[f(x)\overline{g(x)}]_{\xi=0} \\ &= [\widehat{f}(\xi) \star \overline{\widehat{g}(-\xi)}]_{\xi=0} \\ &= [\int_{\mathbb{R}} \widehat{f}(t)\overline{\widehat{g}(t-\xi)}dt]_{\xi=0} \\ &= \int_{\mathbb{R}} \widehat{f}(t)\overline{\widehat{g}(t)}dt.\end{aligned}$$

□

Exercise 2.2. Consider the functions defined on $\mathbb{R}$ by

$$f(x) = \frac{1}{1+x^2}, \quad g(x) = \frac{1}{2-2x+x^2}, \quad h(x) = \frac{x}{(1+x^2)^2},$$

Knowing that $\widehat{f}(\xi) = \pi e^{-2\pi|\xi|}$, determine $\widehat{g}(\xi)$ and $\widehat{h}(\xi)$.

Solution We have $g(x) = \frac{1}{2-2x+x^2} \Rightarrow g(x) = \frac{1}{(x-1)^2+1}$ i.e., $g(x) = (\tau_1 f)(x)$, so $\widehat{g}(\xi) = \mathcal{F}((\tau_1 f)(x))(\xi)$ $\Rightarrow \widehat{g}(\xi) = e^{-2\pi i\xi(1)} \cdot \widehat{f}(\xi)$
i.e.,

$$\widehat{g}(\xi) = e^{-2\pi i\xi}\pi e^{-2\pi|\xi|}.$$

• We have $\frac{d}{dx}(\frac{1}{1+x^2}) = \frac{-2x}{(1+x^2)^2} \Rightarrow h(x) = \frac{-1}{2} f'(x)$, then

$$\begin{aligned}
\widehat{h}(\xi) &= \mathcal{F}(\frac{-1}{2} f'(x))(\xi) \\
&= \frac{-1}{2}\mathcal{F}(f'(x))(\xi) \\
&= \frac{-1}{2}(2\pi i\xi) \cdot \widehat{f}(\xi) \\
&= -i\pi^2 \xi e^{-2\pi|\xi|}.
\end{aligned}$$

Proposition 2.4. *Let $f \in L^1(\mathbb{R})$, then we have*

$$\mathcal{F}(f^{\vee}(x))(\xi) = \widehat{f}(-\xi), \quad \textit{where } f^{\vee}(x) = f(-x).$$

Proof. We have

$$\begin{aligned}
\mathcal{F}(f^{\vee}(x))(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(-x)dx, \quad \text{we put } y = -x \Leftrightarrow dy = -dx \\
&= -\int_{+\infty}^{-\infty} e^{-2\pi i(-y)\xi} f(y)dy \\
&= \int_{-\infty}^{+\infty} e^{-2\pi i y(-\xi)} f(y)dy \\
&= \widehat{f}(-\xi).
\end{aligned}$$

□

Corollary 2.1. *Let $f \in L^1(\mathbb{R})$ be such that $\widehat{f} \in L^1(\mathbb{R})$. Then, for all $x \in \mathbb{R}$ we have*

$$\mathcal{F}(\widehat{f}(\xi))(x) = f(-x).$$

In other words, $\mathcal{F}(\mathcal{F}(f)) = f^{\vee}$ a. e.

2.6 Usual examples

Let $a > 0$ be fixed, c and d be two real numbers such that $c < d$.

Direct calculation : Applying the definition of the Fourier transform we have for all $\xi \in \mathbb{R}$,

(i) $\mathcal{F}(\chi_{[c,d]}(x))(\xi) = \begin{cases} d-c, & \xi = 0 \\ \frac{\sin(\pi(d-c)\xi)}{\pi\xi} e^{-i\pi(c+d)\xi}, & \xi \neq 0 \end{cases}$
in particular $\mathcal{F}(\chi_{[-\frac{a}{2},\frac{a}{2}]}(x))(\xi) = \frac{\sin(\pi a\xi)}{\pi\xi}$.

(ii) $\mathcal{F}(e^{-ax}\chi_{]0,+\infty[}(x))(\xi) = \frac{1}{a+2\pi i\xi}$.

(iii) $\mathcal{F}((1-\frac{2|x|}{a})\chi_{[-\frac{a}{2},\frac{a}{2}]}(x))(\xi) = 2\frac{sin^2(\frac{\pi a\xi}{2})}{\pi^2 a\xi^2}$.

(iv) $\mathcal{F}(e^{ax}\chi_{]-\infty,0[}(x))(\xi) = \frac{1}{a-2\pi i\xi}$.

(v) $\mathcal{F}(e^{-a|x|})(\xi) = \frac{2a}{a^2+4\pi^2\xi^2}$.

(vi) $\mathcal{F}(sign(x)e^{-a|x|})(\xi) = \frac{-4\pi i\xi}{a^2+4\pi^2\xi^2}$.

(vii) $\mathcal{F}(e^{-ax^2})(\xi) = \sqrt{\frac{\pi}{a}}e^{-\frac{\pi^2\xi^2}{a}}$.

(viii) $\mathcal{F}(\frac{1}{ch(ax)})(\xi) = \frac{1}{ch(\frac{\pi^2\xi}{a})}$.

(ix) $\mathcal{F}(\frac{1}{a^2+x^2})(\xi) = \frac{\pi}{a}e^{-2\pi a|\xi|}$.

Theorem 2.1. *Let $f \in L^1(\mathbb{R})$ be such that $\widehat{f}(\xi) = 0$, then $f = 0$ a. e.*

2.7 Resolution of integral equations by the Fourier transform

A Fredholm integral equation of the second kind is an equation of the form

$$\varphi(x) - \int_{-\infty}^{+\infty} k(x,y)\varphi(y)dy = f(x), \tag{2.2}$$

where f and k are given functions, $k(x,y)$ is called the kernel of the integral, and $\varphi(x)$ is the unknown function. To solve it, the kernel must depend on the difference of the arguments, i.e., the equation (2.2) becomes

$$\varphi(x) - \int_{-\infty}^{+\infty} k(x-y)\varphi(y)dy = f(x), \tag{2.3}$$

which implies that

$$\varphi(x) - (k \star \varphi)(x) = f(x).$$

By applying the Fourier transform, we obtain

$$\widehat{\varphi}(\xi) - \widehat{k}(\xi)\cdot\widehat{\varphi}(\xi) = \widehat{f}(\xi), \tag{2.4}$$

with $\widehat{\varphi}(\xi),\ \widehat{k}(\xi),\ \widehat{f}(\xi)$ being the Fourier transforms of $\varphi(x),\ k(x),\ f(x)$ respectively. Under the condition $1-\widehat{k}(\xi) \neq 0$, the equation (2.4) becomes

$$\widehat{\varphi}(\xi) = \frac{\widehat{f}(\xi)}{1-\widehat{k}(\xi)}.$$

Using the Fourier inversion formula, we obtain

$$\varphi(x) = \int_{-\infty}^{+\infty} e^{2\pi i x\xi} \frac{\widehat{f}(\xi)}{1-\widehat{k}(\xi)} d\xi.$$

Exercise 2.3. Solve the following integral equation

$$\varphi(x) - \lambda \int_{\mathbb{R}} e^{-|x-t|}\varphi(t)dt = e^{-|x|},\ \lambda \in \mathbb{R} \text{ et } \lambda > 0, \tag{2.5}$$

knowing that $\mathcal{F}(e^{-a|x|})(\xi) = \frac{2a}{a^2+4\pi^2\xi^2}$.

Solution We put $f(x) = e^{-|x|}$, so the equation (2.5) becomes

$$\varphi(x) - \lambda(f \star \varphi)(x) = f(x).$$

By applying the Fourier transform, we obtain

$$\widehat{\varphi}(\xi) - \lambda(\widehat{f}(\xi) \cdot \widehat{\varphi}(\xi)) = \widehat{f}(\xi),$$

which implies that

$$(1 - \lambda\widehat{f}(\xi))\widehat{\varphi}(\xi) = \widehat{f}(\xi),$$

so we have

$$\widehat{\varphi}(\xi) = \frac{\widehat{f}(\xi)}{1-\lambda\widehat{f}(\xi)}, \quad \text{car } \widehat{f}(\xi) \neq \frac{1}{\lambda}.$$

i.e.,

$$\widehat{\varphi}(\xi) = \frac{2}{(1-2\lambda)+4\pi^2\xi^2}.$$

• If $(1-2\lambda) \leq 0 \Leftrightarrow \lambda \geq \frac{1}{2}$, the function

$$\xi \longmapsto \frac{2}{(1-2\lambda)+4\pi^2\xi^2},$$

is not continuous on $\mathbb{R}$, therefore cannot be the Fourier transform of an integrable function, so the equation (2.5) has no solution.

• If $(1-2\lambda) > 0 \Leftrightarrow \lambda \in]0, \frac{1}{2}[$, the function $\xi \longmapsto \frac{2}{(1-2\lambda)+4\pi^2\xi^2}$ is continuous on $\mathbb{R}$, so we have

$$\begin{aligned} \widehat{\varphi}(\xi) &= \frac{2}{(1-2\lambda)+4\pi^2\xi^2} \\ &= \frac{1}{\sqrt{1-2\lambda}} \cdot \frac{2\sqrt{1-2\lambda}}{(\sqrt{1-2\lambda})^2+4\pi^2\xi^2} \\ &= \frac{1}{\sqrt{1-2\lambda}} \mathcal{F}(e^{-\sqrt{1-2\lambda}|x|})(\xi). \end{aligned}$$

i.e.,

$$\mathcal{F}(\varphi(x))(\xi) = \mathcal{F}(\frac{1}{\sqrt{1-2\lambda}} \cdot e^{-\sqrt{1-2\lambda}|x|})(\xi).$$

Since the Fourier transform is injective on $L^1(\mathbb{R})$, we have

$$\varphi(x) = \frac{1}{\sqrt{1-2\lambda}} \cdot e^{-\sqrt{1-2\lambda}|x|}.$$

2.8 Extension of the Fourier Transform to Square-Integrable Functions

The definition of the Fourier transform given by the formula $\int_{\mathbb{R}} e^{-2\pi i x \cdot \xi} f(x)dx$ is not directly applicable to an arbitrary function in $L^2(\mathbb{R})$. However, this definition is valid when $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and in that case, one can show that $\widehat{f}$ also belongs to $L^2(\mathbb{R})$, and $\|f\|_2 = \left\|\widehat{f}\right\|_2$. This isometry from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ extends to an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$, and this extension allows one to define the Fourier transform (also called the Fourier-Plancherel transform) for any function f of $L^2(\mathbb{R})$.

Theorem 2.2. ***(Plancherel's theorem)*** *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\widehat{f} \in L^2(\mathbb{R})$ and we have*

$$\|f\|_2 = \left\|\widehat{f}\right\|_2.$$

The extension of the Fourier transform to $L^2(\mathbb{R})$ is carried out using the density of $(L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \|\cdot\|_2)$ in $L^2(\mathbb{R})$, and the completion of $L^2(\mathbb{R})$. This is an application of the following result from topology:

Lemma 2.1. *Let E and F be normed vector spaces, with F complete, and let G be a dense subspace of E. If u is a continuous linear map from G into F, then there exists a unique continuous linear extension $\widetilde{u}$ from E into F, and the norm of $\widetilde{u}$ is equal to the norm of u.*

According to Plancherel's theorem (Theorem 2.2) $\mathcal{F}$ is an isometry from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. By applying the above lemma with $E = F = L^2(\mathbb{R})$ and $G = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we obtain the following theorem

Theorem 2.3. ***(Plancherel-Riesz Theorem)*** *There exists a unique automorphism, also denoted by $\mathcal{F}$, of $L^2(\mathbb{R})$ that canonically extends the isometry.*

$$\begin{array}{rcl} L^1(\mathbb{R}) \cap L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ f & \mapsto & \widehat{f} \end{array}$$

Moreover, for every $(f, g) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, we have

1. $\mathcal{F}(\mathcal{F}^{-1}(f)) = \mathcal{F}^{-1}(\mathcal{F}(f)) = f$ *p. p.*

2. $\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$ *Formule de Parseval-Plancherel.*

3. $lim_{A\to+\infty} \|\varphi_A - \mathcal{F}(f)\|_2 = 0$ *où* $\varphi_A(\xi) = \int_{\mathbb{R}} e^{-2\pi ix\cdot\xi} f(x)\chi_{[-A,A]}(x)dx$.

4. $lim_{A\to+\infty} \|\psi_A - f\|_2 = 0$ *où* $\psi_A(x) = \int_{\mathbb{R}} e^{2\pi ix\cdot\xi} \widehat{f}(\xi)\chi_{[-A,A]}(\xi)d\xi$.

Exercise 2.4. We consider the functions defined on $\mathbb{R}$ by

$$f(x) = \frac{\sin(\pi x)}{\pi x} \quad \text{and} \quad g(x) = \sin(\pi x)e^{-\pi x^2}.$$

1) Calculate $\widehat{f}(\xi)$ and deduce the value of the integral $\int_{-\infty}^{+\infty} (\frac{\sin(\pi x)}{\pi x})^2 dx$
2) Calculate $\widehat{g}(\xi)$, Recall that $\mathcal{F}(\pi x e^{-\pi x^2})(\xi) = -i\pi\xi e^{-\pi\xi^2}$.

Solution 1) We know from Example 2.1 that the Fourier transform of the gate function $\pi(x)$ is

$$\widehat{\pi}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

So we have

$$\mathcal{F}(\widehat{\pi}(x))(\xi) = \pi(-\xi),$$

which implies that

$$\mathcal{F}(\frac{\sin(\pi x)}{\pi x})(\xi) = \pi(\xi) \quad \text{because } \pi \text{ is even.}$$

• According to the Parseval-Plancherel formula, we have

$$\begin{aligned} \int_{\mathbb{R}} (\frac{\sin(\pi x)}{\pi x})^2 dx &= \int_{\mathbb{R}} (\pi(\xi))^2 d\xi \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (1)^2 d\xi \\ &= 1. \end{aligned}$$

2) We have

$$\begin{aligned} g(x) &= \sin(\pi x)e^{-\pi x^2} \\ &= \frac{\sin(\pi x)}{\pi x} \cdot \pi x e^{-\pi x^2} \\ &= f(x) \cdot h(x), \quad \text{with } h(x) = \pi x e^{-\pi x^2}. \end{aligned}$$

So we have

$$\begin{aligned}
\widehat{g}(\xi) &= \mathcal{F}(f(x)\cdot h(x))(\xi) \\
&= \widehat{f}(\xi)\star\widehat{h}(\xi) \\
&= \int_{\mathbb{R}}\widehat{f}(\xi-y)\widehat{h}(y)dy \\
&= \int_{\mathbb{R}}\pi(\xi-y)(-i\pi y e^{-\pi y^2})dy \\
&= -i\pi\int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} y e^{-\pi y^2}dy \\
&= -\frac{i}{2}(e^{-\pi\xi^2+\pi\xi-\frac{\pi}{4}} - e^{-\pi\xi^2-\pi\xi-\frac{\pi}{4}}).
\end{aligned}$$

i.e.,

$$\widehat{g}(\xi) = -ie^{-\pi\xi^2-\frac{\pi}{4}}sh(\pi\xi).$$

Theorem 2.4. *1. If $(f,g)\in L^1(\mathbb{R})\times L^2(\mathbb{R})$, so $\mathcal{F}(f\star g)=\mathcal{F}(f)\mathcal{F}(g)$.*

2. If $(f,g)\in L^2(\mathbb{R})\times L^2(\mathbb{R})$, so $\mathcal{F}(f\cdot g)=\mathcal{F}(f)\star\mathcal{F}(g)$.

2.9 Fourier transform in $L^2(\mathbb{R})$ of a derivative

Theorem 2.5. *Let $f\in L^2(\mathbb{R})$*
• *If f is piecewise continuous of class C^1 and such that $f'\in L^2(\mathbb{R})$, then for all $\xi\in\mathbb{R}$,*

$$\mathcal{F}(f'(x))(\xi) = (2\pi i\xi)\widehat{f}(\xi).$$

• *If moreover f is of class C^m piecewise where $m\in\mathbb{N}^\star$ and such that the derivatives $f^{(k)}$ up to order m inclusive are square integrable, then for almost all $\xi\in\mathbb{R}$ and for all $1\le k\le m$, we have*

$$\mathcal{F}(f^{(k)}(x))(\xi) = (2\pi i\xi)^k\widehat{f}(\xi).$$

Exercise 2.5. Is the Fourier transform of the derivative applicable to the function $f(x) = \chi_{[-1,1]}(x)$.

Solution We have $f'(x)=0\ p.\ p.$, so $\widehat{f}'(\xi)=0$.
On the other hand, we have according to the usual example $\mathcal{F}(\chi_{[-1,1]}(x))(\xi)=\frac{\sin(2\pi\xi)}{\pi\xi}$, so

$$(2\pi i\xi)\widehat{f}(\xi) = 2i\sin(2\pi\xi),$$

i.e.,

$$\widehat{f}'(\xi)\neq(2\pi i\xi)\widehat{f}(\xi),\quad \text{because}$$

f is not differentiable at points (1) and (-1).

2.10 Application to the resolution of partial differential equations

2.10.1 Heat equation

Let us consider a homogeneous (very thin) rod of infinite length, isolated from the external environment. At time $t = 0$, the temperature distribution along the rod is given by $u_0(x) = u(0,x)$ for each point $(x \in \mathbb{R})$. We aim to determine its evolution $u(t,x)$, knowing that it satisfies the so-called heat equation.

$$(c)\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad (t,x) \in]0,+\infty[\times\mathbb{R} \\ u(0,x) = u_0(x), \quad x \in \mathbb{R} \end{cases}.$$

• We assume that $u_0 \in L^1(\mathbb{R})$, and we look for a function $u(t,x) \in C^{1,2}(]0,+\infty[\times\mathbb{R})$.
• We assume that for fixed $t > 0$, we have

$$\int_{\mathbb{R}} |u(t,x)|\,dx < +\infty,\ \int_{\mathbb{R}} \left|\frac{\partial u}{\partial t}(t,x)\right| dx < +\infty,\ \int_{\mathbb{R}} \left|\frac{\partial^2 u}{\partial x^2}(t,x)\right| dx < +\infty,$$

so that the functions $x \mapsto u(t,x)$, $x \mapsto \frac{\partial u}{\partial t}(t,x)$, $x \mapsto \frac{\partial^2 u}{\partial x^2}(t,x)$ have, for each fixed value of $t > 0$, a Fourier transform with respect to the spatial variable x.
• Moreover, we assume that for $t > 0$, we have

$$\int_{\mathbb{R}} \frac{\partial u}{\partial t}(t,x)e^{-2\pi i x\cdot\xi}dx = \frac{\partial}{\partial t}(\int_{\mathbb{R}} u(t,x)e^{-2\pi i x\cdot\xi}dx).$$

We put for $t \geq 0$ fixed

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi}u(t,x)dx.$$

Applying the Fourier transform with respect to the space variable x, we obtain

$$\mathcal{F}(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2})(\xi) = 0,$$

which implies that

$$\mathcal{F}(\frac{\partial u}{\partial t})(\xi) - \mathcal{F}(\frac{\partial^2 u}{\partial x^2})(\xi) = 0,$$

so we have

$$\int_{\mathbb{R}} e^{-2\pi i x\cdot\xi}\frac{\partial u}{\partial t}(t,x)dx - (2\pi i\xi)^2\widehat{u}(t,\xi) = 0,$$

i.e.,

$$\frac{\partial}{\partial t}\widehat{u}(t,\xi) + 4\pi^2\xi^2\widehat{u}(t,\xi) = 0. \tag{2.6}$$

Thus, $\forall \xi \in \mathbb{R}$ fixed, $\widehat{u}$ is the solution of the differential equation with respect to time t, therefore

$$\widehat{u}(t,\xi) = ce^{-4\pi^2\xi^2 t}.$$

For $t = 0 \Rightarrow \widehat{u}(0,\xi) = c = \widehat{u}_0(\xi)$, so the solution is

$$\widehat{u}(t,\xi) = \widehat{u}_0(\xi)e^{-4\pi^2\xi^2 t}.$$

The Fourier inversion formula gives

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}(\widehat{u}_0(\xi)e^{-4\pi^2\xi^2 t})(x) \\ &= \mathcal{F}^{-1}(\widehat{u}_0(\xi))(x) \star \mathcal{F}^{-1}(e^{-4\pi^2\xi^2 t})(x) \\ &= u_0(x) \star \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \quad \text{car } \mathcal{F}(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}})(\xi) = e^{-4\pi^2\xi^2 t} \\ &= (u_0 \star \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}})(x), \end{aligned}$$

i.e.,

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}u_0(y)dy.$$

Exercise 2.6. Let f be the function defined on $\mathbb{R}$ by $f(x) = e^{-\pi x^2}$
1) Check that f is the solution to the differential equation

$$f'(x) + 2\pi x f(x) = 0 \tag{2.7}$$

2) By applying the Fourier transform to (2.7), show that $\widehat{f}(\xi)$ is a solution to a first-order differential equation $\widehat{(2.7)}$ that we will determine.
3) Solve $\widehat{(2.7)}$ and determine $\widehat{f}(\xi)$, knowing that $\int_{\mathbb{R}} e^{-\pi x^2}dx = 1$.

Solution 1) We have

$$\begin{aligned} f'(x) + 2\pi x f(x) &= -2\pi x e^{-\pi x^2} + 2\pi x e^{-\pi x^2} \\ &= 0. \end{aligned}$$

2) Applying the Fourier transform, we obtain

$$\mathcal{F}(f'(x) + 2\pi x f(x))(\xi) = 0,$$

which implies that

$$\mathcal{F}(f'(x))(\xi) + 2\pi\mathcal{F}(xf(x))(\xi) = 0, \quad \text{because } \mathcal{F} \text{ is linear}$$

i.e.,

$$(2\pi i\xi)\widehat{f}(\xi) + 2\pi(\frac{-1}{2\pi i}\frac{d}{d\xi}\widehat{f}(\xi)) = 0,$$

then the equation $\widehat{(2.7)}$ becomes as follows

$$\frac{d}{d\xi}\widehat{f}(\xi) + 2\pi\xi\widehat{f}(\xi) = 0.$$

3) We have $\frac{d}{d\xi}\widehat{f}(\xi) + 2\pi\xi\widehat{f}(\xi) = 0 \Rightarrow \frac{d}{d\xi}\widehat{f}(\xi) = -2\pi\xi\widehat{f}(\xi)$, which implies that $\frac{d\widehat{f}(\xi)}{\widehat{f}(\xi)} = -2\pi\xi d\xi \Rightarrow \int \frac{d\widehat{f}(\xi)}{\widehat{f}(\xi)} = \int -2\pi\xi d\xi \Rightarrow \ln(\frac{\widehat{f}(\xi)}{c}) = -\pi\xi^2$,
i.e.,

$$\widehat{f}(\xi) = ce^{-\pi\xi^2}.$$

For $\xi = 0$, we have $\widehat{f}(0) = c$ and $\widehat{f}(0) = \int_{\mathbb{R}} f(x)dx$,
which implies that

$$\begin{aligned} c &= \int_{\mathbb{R}} e^{-\pi x^2} dx \\ &= 1, \end{aligned}$$

i.e.,

$$\widehat{f}(\xi) = e^{-\pi\xi^2}.$$

CHAPTER 3

LAPLACE TRANSFORM

One of the most efficient methods for solving certain differential equations is to use the Laplace transform . The Laplace transform transforms functions $f(x)$ into other functions $F(s)$. We write

$$F = \mathcal{L}(f) \quad \text{or} \quad F(s) = \mathcal{L}(f(x))(s).$$

The inverse Laplace transform transforms $F(s)$ into $f(x)$. We write

$$f = \mathcal{L}^{-1}(F) \quad \text{or} \quad f(x) = \mathcal{L}^{-1}(F(s))(x).$$

We will see later on which functions these transforms are defined. The essential property is that, under certain conditions,

$$\mathcal{L}(f'(x))(s) = s \cdot F(s).$$

Thus, differential equations become algebraic equations.

3.1 C_L Functions

The class of real functions C_L is formed by causal functions, piecewise continuous, and of exponential order.

• A function is causal if it is zero for $x < 0,\ f(x) = 0$ if $x < 0$.

• It continues piecewise if it only has points of discontinuity of the first kind (having a left limit and a right limit).

• It is of exponential order if it is bounded by an exponential, i.e., if there exist real constants $M \geq 0$ and α such that

$$|f(x)| \leq Me^{\alpha x}, \quad \forall x \geq x_0.$$

• The usual functions $\sin(\omega x)$, x^2, e^x are not causal, one way to create causal functions is to use the Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

For example the function $f(x) = e^x$ is not a causal function, but if we multiply it by $H(x)$, we have

$$\begin{aligned} f(x) &= e^x H(x) \\ &= \begin{cases} e^x, & x \geq 0 \\ 0, & x < 0 \end{cases}. \end{aligned}$$

3.2 Definition of Laplace transform

Definition 3.1. The Laplace transform of a function of C_L is defined by

$$\mathcal{L}(f(x))(s) = F(s) = \int_0^{+\infty} e^{-sx} f(x)dx,$$

s is here a complex variable (frequency) and $F(s)$ a complex function.

Remarks 3.1. 1) $F(s)$ defined by an improper integral which does not always converge if $f \notin C_L$.
2) If f is discontinuous at 0, the lower bound of the integral should be denoted 0^+.

Exercise 3.1. Calculate the Laplace transform of the following functions

$$1)\ f(x) = H(x)e^{2x} = \begin{cases} e^{2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \qquad 2)\ g(x) = H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Solution 1) Let $F(s) = \mathcal{L}(f(x))(s)$, so

$$\begin{aligned} F(s) &= \int_0^{+\infty} e^{-sx} f(x)dx \\ &= \int_0^{+\infty} e^{(2-s)x} dx \\ &= \frac{1}{2-s}[e^{(2-s)x}]_0^{+\infty}. \end{aligned}$$

Let $s = \alpha + i\beta \Rightarrow e^{(2-s)x} = e^{(2-\alpha)x} e^{-i\beta x}$, so

$$lim_{x \to +\infty} \left| e^{(2-s)x} \right| = lim_{x \to +\infty} e^{(2-\alpha)x} = 0 \text{ if } \alpha > 2, \text{ so}$$

$$\begin{aligned} F(s) &= \frac{1}{2-s}[e^{(2-s)x}]_0^{+\infty} \\ &= \frac{1}{2-s}(0-1) \\ &= \frac{1}{s-2}, \end{aligned}$$

i.e.,

$$F(s) = \frac{1}{s-2}, \quad Re(s) > 2.$$

2) Let $G(s) = \mathcal{L}(g(x))(s)$, so

$$\begin{aligned} G(s) &= \int_0^{+\infty} e^{-sx} H(x) dx \\ &= \int_0^{+\infty} e^{-sx} dx \\ &= \frac{-1}{s}[e^{-sx}]_0^{+\infty}. \end{aligned}$$

Let $s = \alpha + i\beta \Rightarrow e^{-sx} = e^{-\alpha x} e^{-i\beta x}$, so

$$lim_{x \to +\infty} \left| e^{-sx} \right| = lim_{x \to +\infty} e^{-\alpha x} = 0 \text{ if } \alpha > 0, \text{ so}$$

$$\begin{aligned} G(s) &= \frac{-1}{s}[e^{-sx}]_0^{+\infty} \\ &= \frac{-1}{s}(0-1) \\ &= \frac{1}{s}, \end{aligned}$$

i.e.,

$$G(s) = \frac{1}{s}, \quad Re(s) > 0.$$

Theorem 3.1. *Let f be a function defined on $\mathbb{R}$ and such that*

(i) $f(x) = 0, \ \forall x < 0$*,*

(ii) *f is piecewise continuous on $[0, +\infty[$,*

(iii) *there exist constants $M \geq 0$ and r such that*

$$\forall x \geq x_0; |f(x)| \leq Me^{rx}.$$

So, the Laplace transform of f exists for all $Re(s) > r$.

Proof. We have

$$\int_0^{+\infty} e^{-sx} f(x)dx = \int_0^{x_0} e^{-sx} f(x)dx + \int_{x_0}^{+\infty} e^{-sx} f(x)dx.$$

The integral $\int_0^{x_0} e^{-sx} f(x)dx$ exists because f is piecewise continuous. As for the other integral, note that

$$\begin{aligned} \left|e^{-sx} f(x)\right| &= \left|e^{-(\alpha+i\beta)x} f(x)\right| \\ &= e^{-\alpha x} \left|f(x)\right| \\ &\leq Me^{-(\alpha-r)x}. \end{aligned}$$

Now, the integral $\int_{x_0}^{+\infty} Me^{-(\alpha-r)x}dx$ converges because $Re(s) = \alpha > r$. Therefore, by the comparison test for improper integrals, the integral $\int_{x_0}^{+\infty} \left|e^{-sx} f(x)\right| dx$ also converges, which implies that $\int_{x_0}^{+\infty} e^{-sx} f(x)dx$ exists. Consequently, the integral $\int_0^{+\infty} e^{-sx} f(x)dx$ exists in the half-plane

$$\{s \in \mathbb{C} : Re(s) > r\}.$$

□

Exercise 3.2. Compute the Laplace transform of

$$1)\ f(x) = \begin{cases} x^n, & x \geq 0 \\ 0, & x < 0 \end{cases}, \ n \in \mathbb{N}.$$

Solution We have

$$F(s) = \int_0^{+\infty} e^{-sx} \cdot x^n dx = I_n,$$

Using integration by parts, we obtain

$$\begin{aligned} I_n &= [-\frac{x^n}{s}e^{-sx}]_0^{+\infty} + \frac{n}{s}\int_0^{+\infty} x^{n-1} \cdot e^{-sx}dx \\ &= \frac{n}{s}\int_0^{+\infty} x^{n-1} \cdot e^{-sx}dx, \quad \text{car } lim_{x\to+\infty}(-\frac{x^n}{s}e^{-sx}) = 0, \text{ si } \alpha > 0 \text{ avec } s = \alpha + i\beta, \end{aligned}$$

i.e.,

$$I_n = \frac{n}{s} \cdot I_{n-1}.$$

By recurrence, we obtain

$$I_n = \frac{n}{s} \cdot \frac{n-1}{s} \times ... \times \frac{1}{s} \cdot I_0,$$

and

$$\begin{aligned} I_0 &= \int_0^{+\infty} e^{-sx} dx \\ &= \frac{1}{s}, \text{ si } \alpha > 0. \end{aligned}$$

So

$$I_n = \frac{n}{s} \times \frac{n-1}{s} \times ... \times \frac{1}{s} \times \frac{1}{s}, \quad Re(s) > 0,$$

i.e.,

$$F(s) = \frac{n!}{s^{n+1}}, \quad Re(s) > 0.$$

3.3 Properties of the Laplace Transform

1) Linearity The Laplace transform is a linear operator. More precisely, for all $\alpha, \beta \in \mathbb{C}$, and for all functions f, g with respective abscissas of summability r, σ, we have

$$\mathcal{L}(\alpha f(x) + \beta g(x))(s) = \alpha F(s) + \beta G(s),$$

where $F(s) = \mathcal{L}(f(x))(s),\ G(s) = \mathcal{L}(g(x))(s)$, and $Re(s) > \max(r, \sigma)$.

Proof. Indeed, if the functions f and g admit Laplace transforms

$$\mathcal{L}(f(x))(s) = F(s) = \int_0^{+\infty} e^{-sx} f(x) dx, \quad \mathcal{L}(g(x))(s) = G(s) = \int_0^{+\infty} e^{-sx} g(x) dx,$$

then we have

$$\begin{aligned} \mathcal{L}(\alpha f(x) + \beta g(x))(s) &= \int_0^{+\infty} e^{-sx} (\alpha f(x) + \beta g(x)) dx \\ &= \alpha \int_0^{+\infty} e^{-sx} f(x) dx + \beta \int_0^{+\infty} e^{-sx} g(x) dx \\ &= \alpha F(s) + \beta G(s). \end{aligned}$$

• If the abscissas of summability of f and g are r and σ respectively, then the domain of summability on which $\alpha f + \beta g$ is defined is $\{s \in \mathbb{C} : Re(s) > \max(r, \sigma)\}$. □

2) Translation If $\mathcal{L}(f(x))(s) = F(s)$ with $Re(s) > r$, then

$$\mathcal{L}(\tau_a f)(s) = e^{-as} F(s), \quad Re(s) > r.$$

Where $(\tau_a f)(x) = f(x - a)$.

Proof. We put $g(x) = \begin{cases} f(x-a), & x \geq a \\ 0, & x < a \end{cases}$.

We have

$$\begin{aligned} \mathcal{L}(g(x))(s) &= \int_0^{+\infty} e^{-sx} g(x)dx \\ &= \int_0^{a} e^{-sx} g(x)dx + \int_a^{+\infty} e^{-sx} g(x)dx \\ &= \int_a^{+\infty} e^{-sx} f(x-a)dx, \end{aligned}$$

we put $x - a = t \Leftrightarrow dt = dx$, so

$$\begin{aligned} \mathcal{L}(g(x))(s) &= e^{-as} \int_0^{+\infty} e^{-st} f(t)dt \\ &= e^{-as} F(s). \end{aligned}$$

□

Exercise 3.3. Find the Laplace transform of

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < a \\ 2, & x > a \end{cases}.$$

Solution We have $f(x) = H(x) + H(x-a)$, where is the Heaviside function, and let $F(s) = \mathcal{L}(f(x))(s)$, so $F(s) = \mathcal{L}(H(x))(s) + \mathcal{L}(\tau_a H)(s)$,
i.e.,

$$F(s) = \frac{1}{s} + e^{-as} \cdot \frac{1}{s}, \quad Re(s) > 0.$$

3) Property If $F(s) = \mathcal{L}(f(x))(s)$, then

$$\mathcal{L}(f(x)e^{-\alpha x})(s) = F(s+\alpha), \quad Re(s+\alpha) > r.$$

Proof. We have

$$\begin{aligned} \mathcal{L}(f(x)e^{-\alpha x})(s) &= \int_0^{+\infty} e^{-sx} f(x) e^{-\alpha x} dx \\ &= \int_0^{+\infty} e^{-(\alpha+s)x} f(x)dx \\ &= F(s+\alpha). \end{aligned}$$

□

4) Change of scale If $\mathcal{L}(f(x))(s) = F(s)$, then

$$\mathcal{L}(f(\lambda x))(s) = \frac{1}{\lambda}F(\frac{s}{\lambda}),\ \lambda > 0.$$

Proof. We have

$$\begin{aligned}\mathcal{L}(f(\lambda x))(s) &= \int_0^{+\infty} e^{-sx} f(\lambda x)dx, \quad \text{on pose } t = \lambda x \Leftrightarrow dt = \lambda dx \\ &= \frac{1}{\lambda}\int_0^{+\infty} e^{-\frac{s}{\lambda}t} f(t)dt \\ &= \frac{1}{\lambda}F(\frac{s}{\lambda}).\end{aligned}$$

□

4) Complex conjugate If $\mathcal{L}(f(x))(s) = F(s)$, then

$$\mathcal{L}(\overline{f(x)})(s) = \overline{F}(\overline{s})$$

Proof. We have

$$\begin{aligned}\mathcal{L}(\overline{f(x)})(s) &= \int_0^{+\infty} e^{-sx}\overline{f(x)}dx \\ &= \overline{\int_0^{+\infty} e^{-\overline{s}x} f(x)dx} \\ &= \overline{F}(\overline{s}).\end{aligned}$$

□

Proposition 3.1. *The Laplace transform of a locally integrable function f, is a holomorphic function in the domain of summability $\{s \in \mathbb{C} : Re(s) > r\}$ and we have the formula*

$$F^{(n)}(s) = \int_0^{+\infty} (-x)^n f(x)e^{-sx}dx = (-1)^n\mathcal{L}(x^n f(x))(s).$$

Exercise 3.4. Determine the Laplace transform of x^n, using the previous proposition.

Solution We have

$$\mathcal{L}(x^n f(x))(s) = (-1)^n F^{(n)}(s), \quad \text{ici } f(x) = 1,$$

so we have

$$\mathcal{L}(x^n)(s) = (-1)^n(\frac{1}{s})^{(n)}, \quad Re(s) > 0.$$

Using the successive derivative of order n, we can demonstrate by recurrence that

$$(\frac{1}{s})^{(n)} = \frac{n!}{(-1)^n s^{n+1}},$$

i.e.,

$$\mathcal{L}(x^n)(s) = \frac{n!}{s^{n+1}}, \quad Re(s) > 0.$$

3.4 Transform of the derivative

Theorem 3.2. *If f' is piecewise continuous on all closed sets $[0, x_0]$ and if $\mathcal{L}(f(x))(s) = F(s)$ and if there exist $M > 0$ and r such that $|f(x)| \leq Me^{rx}$, $\forall x \geq x_0$ then*

$$\mathcal{L}(f'(x))(s) = sF(s) - f(0^+), \quad Re(s) > r.$$

Proof. We have

$$\mathcal{L}(f'(x))(s) = \int_0^{+\infty} e^{-sx} f'(x)dx,$$

by integrating by parts we obtain

$$\mathcal{L}(f'(x))(s) = [e^{-sx} f(x)]_0^{+\infty} + s\int_0^{+\infty} e^{-sx} f(x)dx.$$

As $lim_{x\to+\infty} e^{-sx} f(x) = 0$, because

$$\begin{aligned} lim_{x\to+\infty} \left|e^{-sx} f(x)\right| &= lim_{x\to+\infty} e^{-\alpha x} \left|f(x)\right|, \quad s = \alpha + i\beta \\ &\leq lim_{x\to+\infty} Me^{(r-\alpha)x} = 0, \quad \text{with } Re(s) > r. \end{aligned}$$

So, $[e^{-sx} f(x)]_0^{+\infty} = -f(0^+$.
$f(0^+)$ representing the right limit of $f(x)$ when $x \to 0$.
Hence

$$\mathcal{L}(f'(x))(s) = sF(s) - f(0^+), \quad Re(s) > r.$$

□

Generalization If f'' also satisfies the assumptions of the theorem, then we have

$$\begin{aligned} \mathcal{L}(f''(x))(s) &= s\mathcal{L}(f'(x))(s) - f'(0^+) \\ &= s(sF(s) - f(0^+)) - f'(0^+), \end{aligned}$$

hence

$$\mathcal{L}(f''(x))(s) = s^2F(s) - sf(0^+) - f'(0^+).$$

• We can demonstrate by recurrence that

$$\mathcal{L}(f^{(n)}(x))(s) = s^nF(s) - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - ... - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+).$$

Special case If $f(0^+) = f'(0^+) = ... = f^{(n-1)}(0^+) = 0$, we have

$$\mathcal{L}(f^{(n)}(x))(s) = s^nF(s).$$

Remark 3.1. In general, if $f(x)$ is discontinuous at points $x_1,\ x_2, ...,\ x_n$, then

$$\mathcal{L}(f'(x))(s) = sF(s) - f(0^+) - \sum_{k=1}^{n} e^{-sx_k}(f(x_k^+) - f(x_k^-)).$$

Proposition 3.2. *If* $\mathcal{L}(f(x))(s) = F(s)$, *then*

$$\mathcal{L}(\int_0^x f(t)dt)(s) = \frac{F(s)}{s}, \quad Re(s) > \max(0, r).$$

Proof. Let $g(x) = \int_0^x f(t)dt$. From the previous result, we have

$$\begin{aligned}\mathcal{L}(g'(x))(s) &= s\mathcal{L}(g(x))(s) - g(0^+)\\ &= s\mathcal{L}(g(x))(s),\end{aligned}$$

Since $g(0) = 0$, and $g'(x) = f(x)$, it follows that $\mathcal{L}(g(x))(s) = \mathcal{L}(f(x))(s)$. We then deduce that $s\mathcal{L}(g(x))(s) = \mathcal{L}(f(x))(s)$.
i.e.,

$$\begin{aligned}\mathcal{L}(\int_0^x f(t)dt)(s) &= \frac{\mathcal{L}(f(x))(s)}{s}\\ &= \frac{F(s)}{s}.\end{aligned}$$

□

3.5 Inverse Laplace transform

Let $F(s)$ be the Laplace transform of a function $f(x)$. We call the inverse Laplace transform, or the original of $F(s)$, the function $f(x)$, and we denote $f(x) = \mathcal{L}^{-1}(F(s))(x)$.

Example 3.1. 1) $\mathcal{L}^{-1}(\frac{1}{s^2})(x) = xH(x)$.
2) $\mathcal{L}^{-1}(\frac{s}{s^2+4})(x) = \cos(2x)H(x)$, because $\mathcal{L}(\cos(ax))(s) = \frac{s}{s^2+a^2}$.

• It can be shown that if the functions f considered satisfy the properties stated at the beginning of the chapter, i.e.,
• A causal function.
• Piecewise continuous on every closed interval $[0, x_0]$.
• Of exponential order.
Then the original function $f(x)$ of a given function $F(s)$ is unique on any subset where it is continuous.
The search for the original leads to the study of the properties of the mapping $F(s) \overset{\mathcal{L}^{-1}}{\to} f(x)$, called the inverse Laplace transform.

3.6 Properties of the inverse Laplace transform

1) Linearity Since the inverse of a linear operator is also linear, we have

$$\mathcal{L}^{-1}(\alpha F(s) + \beta G(s))(x) = \alpha\mathcal{L}^{-1}(F(s))(x) + \beta\mathcal{L}^{-1}(G(s))(x).$$

• In general, to find the original function corresponding to a rational function $F(s) = \frac{N(s)}{D(s)}$, one uses its partial fraction decomposition.

Exercise 3.5. Find the original of $F(s) = \frac{s+1}{s^2(s^2+4)}$.

Solution The decomposition of $F(s)$ is written as

$$\frac{s+1}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s} + \frac{cs+d}{s^2+4},$$

and the computation yields $A = \frac{1}{4}$, $B = \frac{1}{4}$, $c = \frac{-1}{4}$, $d = \frac{-1}{4}$,
Hence,

$$f(x) = \frac{1}{4}\mathcal{L}^{-1}(\frac{1}{s^2})(x) + \frac{1}{4}\mathcal{L}^{-1}(\frac{1}{s})(x) - \frac{1}{4}\mathcal{L}^{-1}(\frac{s}{s^2+4})(x) - \frac{1}{4}\mathcal{L}^{-1}(\frac{1}{s^2+4})(x),$$

i.e.,

$$f(x) = (\frac{1}{4} + \frac{1}{4}x - \frac{1}{4}\cos(2x) - \frac{1}{8}\sin(2x))H(x).$$

2) Original of $F(as)$**,** $a > 0$ Let $f(x)$ be the original of $F(s)$, i.e., $f(x) = \mathcal{L}^{-1}(F(s))(x)$, we have

$$\begin{aligned} F(as) &= \int_0^{+\infty} e^{-asx} f(x)dx, \quad y = ax \Leftrightarrow dy = adx \\ &= \frac{1}{a}\int_0^{+\infty} e^{-sy} f(\frac{y}{a})dy, \end{aligned}$$

hence,

$$\begin{aligned} F(as) &= \frac{1}{a}\mathcal{L}(f(\frac{x}{a}))(s) \\ &= \mathcal{L}(\frac{1}{a}f(\frac{x}{a}))(s), \end{aligned}$$

i.e.,

$$\mathcal{L}^{-1}(F(as))(x) = \frac{1}{a}f(\frac{x}{a}).$$

3) Original of $F(s+a)$ Let $f(x) = \mathcal{L}^{-1}(F(s))(x)$. Then,

$$\begin{aligned} F(s+a) &= \int_0^{+\infty} e^{-(s+a)x} f(x)dx \\ &= \int_0^{+\infty} e^{-sx}(e^{-ax}f(x))dx \\ &= \mathcal{L}(e^{-ax}f(x))(s), \end{aligned}$$

hence,

$$\mathcal{L}^{-1}(F(s+a))(x) = e^{-ax}f(x).$$

Example 3.2. 1) $\mathcal{L}^{-1}(\frac{s+a}{(s+a)^2+a^2})(x) = e^{-ax}\cos(ax),\ x \geq 0$
2) $\mathcal{L}^{-1}(\frac{a}{(s+a)^2+a^2})(x) = e^{-ax}\sin(ax),\ x \geq 0$

Exercise 3.6. Find the original of $F(s) = \frac{s}{s^2+s+1}$

Solution We can write

$$\begin{aligned} \frac{s}{s^2+s+1} &= \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} \\ &= \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}, \end{aligned}$$

therefore,

$$f(x) = \mathcal{L}^{-1}(\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}})(x) - \frac{1}{2}\mathcal{L}^{-1}(\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}})(x),$$

i.e.,

$$f(x) = e^{-\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x),\ x \geq 0.$$

4) Original of $F(s) \times G(s)$

Theorem 3.3. *If* $\mathcal{L}^{-1}(F(s))(x) = f(x)$ *and* $\mathcal{L}^{-1}(G(s))(x) = g(x)$, *then*

$$\mathcal{L}^{-1}(F(s) \times G(s))(x) = \int_0^x f(t)g(x-t)dt,$$

the integral $\int_0^x f(t)g(x-t)dt$ *is called the convolution product of* f *by* g *and is denoted* $(f \star g)(x)$

• We easily verify that $(f \star g)(x) = (g \star f)(x)$.

Exercise 3.7. Find the original of $F(s) = \frac{1}{s^2(s+1)}$

Solution Let $f(x) = \mathcal{L}^{-1}(F(s))(x)$, we have

$$\mathcal{L}^{-1}(\frac{1}{s^2})(x) = x \quad \text{et} \quad \mathcal{L}^{-1}(\frac{1}{s+1})(x) = e^{-x},$$

hence

$$\begin{aligned} \mathcal{L}^{-1}(\frac{1}{s^2(s+1)})(x) &= \mathcal{L}^{-1}(\frac{1}{s^2} \cdot \frac{1}{s+1})(x) \\ &= \int_0^x te^{-(x-t)}dt, \end{aligned}$$

i.e.,

$$f(x) = e^{-x} + x - 1, \ x \geq 0.$$

3.7 Table of Some Common Functions

The table below gives some Laplace transforms of common functions.

$f(x) = \mathcal{L}^{-1}(F(s))(x)$	$F(s) = \mathcal{L}(f(x))(s)$
$H(x)$	$\frac{1}{s}, \quad Re(s) > 0$
e^{ax}	$\frac{1}{s-a}, \quad Re(s) > a$
$\sin(ax)$	$\frac{a}{s^2+a^2}, \quad Re(s) > 0$
$\cos(ax)$	$\frac{s}{s^2+a^2}, \quad Re(s) > 0$
$x^\alpha, \ \alpha > -1$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad Re(s) > 0$
$x^n, \ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad Re(s) > 0$
$f(ax)$	$\frac{1}{a}F(\frac{s}{a})$
$e^{-ax}f(x)$	$F(s+a)$
$f(x-a)H(x-a)$	$e^{-as}F(s)$

3.8 Application of the Laplace transform to differential equations

Let us consider the linear differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + ... + a_0 y(x) = f(x).$$

Let $\mathcal{L}(y(x))(s) = Y(s)$. Then,

$$\mathcal{L}(y'(x))(s) = sY(s) - y(0) \quad \text{et} \quad \mathcal{L}(y''(x))(s) = s^2Y(s) - sy(0) - y'(0),$$

and more generally,

$$\mathcal{L}(y^{(n)}(x))(s) = s^nY(s) - s^{n-1}y(0) - ... - y^{(n-1)}(0).$$

By applying the Laplace transform to the above differential equation, and using linearity, we obtain

$$(a_ns^n + a_{n-1}s^{n-1} + ... + a_0)Y(s) + \phi(s) = F(s),$$

where $\phi(s)$ is a polynomial of degree at most $(n-1)$, involving the initial conditions $y(0), y'(0), ..., y^{(n-1)}(0)$. It follows that

$$Y(s) = \frac{F(s) - \phi(s)}{a_ns^n + a_{n-1}s^{n-1} + ... + a_0},$$

and consequently, by applying the inverse Laplace transform,

$$y(x) = \mathcal{L}^{-1}(Y(s))(x).$$

Exercise 3.8. Find the solution to the following differential equation

$$y'' - 2y' + y = xe^x, \quad y(0) = 1,\ y'(0) = 0. \tag{3.1}$$

Solution Let $\mathcal{L}(y(x))(s) = Y(s)$, so $\mathcal{L}(y'(x))(s) = sY(s) - y(0) = sY(s) - 1$, $\mathcal{L}(y''(x))(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$.
Applying the Laplace transform to the equation (3.1), we obtain

$$\mathcal{L}(y''(x))(s) - 2\mathcal{L}(y'(x))(s) + \mathcal{L}(y(x))(s) = \mathcal{L}(xe^x)(s),$$

which implies that

$$s^2Y(s) - s - 2sY(s) + 2 + Y(s) = \frac{1}{(s-1)^2},$$

i.e.,

$$Y(s) = \frac{1}{(s-1)^4} - \frac{1}{(s-1)^2} + \frac{1}{s-1},$$

applying the inverse Laplace transform, we obtain

$$y(x) = e^x(\frac{1}{6}x^3 - x + 1).$$

3.9 Solving integral equations

The Laplace transform allows the study of a large number of integral equations.

• A Volterra integral equation of the second kind is an equation of the form

$$\varphi(x) - \int_0^x k(x,t)\varphi(t)dt = g(x),$$

where $g,\ k$ are known functions and φ an unknown function, the function k is the kernel of this equation. We consider the case where the kernel depends only on the difference $x-t$, i.e., $k(x,t) = k(x-t)$ with k with support in $\mathbb{R}_+$. Let F, G and K be the Laplace transforms of $\varphi,\ g$ and k, respectively. Applying the Laplace transform to both sides of the above equation, we obtain

$$\mathcal{L}(\varphi(x))(s) - \mathcal{L}\left(\int_0^x k(x-t)\varphi(t)dt\right)(s) = \mathcal{L}(g(x))(s),$$

so we have

$$\mathcal{L}(\varphi(x))(s) - \mathcal{L}\left((k \star \varphi)(x)\right)(s) = \mathcal{L}(g(x))(s),$$

hence

$$F(s) - K(s)F(s) = G(s),$$

hence

$$F(s) = \frac{G(s)}{1-K(s)}, \quad K(s) \neq 1.$$

The original $\varphi(x)$ of $F(s)$ is the solution to the integral equation.

Exercise 3.9. Determine the solution to the following integral equation

$$\varphi(x) - \int_0^x \sin(x-t)\varphi(t)dt = x^2. \tag{3.2}$$

Solution Let $F(s) = \mathcal{L}(\varphi(x))(s)$.

The equation (3.2) is written in the form

$$\varphi(x) - \sin(x) \star \varphi(x) = x^2, \quad x \geq 0.$$

Applying the Laplace transform to both sides, we have

$$F(s) - \frac{1}{s^2+1}F(s) = \frac{2}{s^3}, \quad Re(s) > 0,$$

i.e.,

$$F(s) = \frac{2}{s^5} + \frac{2}{s^3}, \quad Re(s) > 0.$$

By consequently

$$\varphi(x) = \mathcal{L}^{-1}(F(s))(x) = \frac{1}{12}x^4 + x^2.$$

3.10 Solving partial differential equations

The Laplace transform method can be used to solve certain partial differential equations, as shown in the following example

Example 3.3. Solve the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \text{ with}$$

$$u(x,0) = \sin(x),\ u(0,t) = 0,\ u(\pi,t) = 0,\ 0 < x < \pi,\ t > 0.$$

Solution Let $U(x,s) = \mathcal{L}(u(x,t))(s)$ be the Laplace transform of $u(x,t)$. We have

$$\mathcal{L}(\frac{\partial u(x,t)}{\partial t})(s) = \mathcal{L}(\frac{\partial^2 u(x,t)}{\partial x^2})(s). \tag{3.3}$$

We have $\mathcal{L}(\frac{\partial u}{\partial t})(s) = \int_0^{+\infty} e^{-st}\frac{\partial u(x,t)}{\partial t}dt$ and since $\mathcal{L}(f'(x))(s) = sF(s) - f(0)$, then

$$\mathcal{L}(\frac{\partial u}{\partial t})(s) = sU(x,s) - u(x,0) = sU(x,s) - \sin(x),$$

and

$$\begin{aligned}
\mathcal{L}(\frac{\partial^2 u}{\partial x^2})(s) &= \mathcal{L}(\frac{\partial v}{\partial x})(s),\ v = \frac{\partial u}{\partial x} \\
&= \int_0^{+\infty} e^{-st}\frac{\partial v(x,t)}{\partial x}dt \\
&= \frac{\partial}{\partial x}\int_0^{+\infty} e^{-st}v(x,t)dt \\
&= \frac{\partial}{\partial x}\int_0^{+\infty} e^{-st}\frac{\partial u(x,t)}{\partial x}dt \\
&= \frac{\partial^2}{\partial x^2}\int_0^{+\infty} e^{-st}u(x,t)dt \\
&= \frac{\partial^2 U(x,s)}{\partial x^2},
\end{aligned}$$

then the equation (3.3) becomes

$$\frac{\partial^2 U(x,s)}{\partial x^2} = sU(x,s) - \sin(x),$$

i.e.,

$$\frac{\partial^2 U(x,s)}{\partial x^2} - sU(x,s) = -\sin(x).$$

This is a second-order differential equation with constant coefficient, so the solution is

$$U(x,s) = y_H(x) + y_p(x).$$

The homogeneous equation is

$$(H): \ \frac{\partial^2 U(x,s)}{\partial x^2} - sU(x,s) = 0,$$

the characteristic equation is $r^2 - s = 0 \Rightarrow r_1 = \sqrt{s},\ r_2 = -\sqrt{s}$, so we have

$$y_H(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x},\ c_1,\ c_2 \in \mathbb{R} \text{ and } y_p(x) = \frac{\sin(x)}{1+s},$$

then the general solution is

$$U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{\sin(x)}{1+s},\ c_1,\ c_2 \in \mathbb{R}.$$

We have $U(0,s) = c_1 + c_2$ and $U(\pi,s) = c_1 e^{\sqrt{s}\pi} + c_2 e^{-\sqrt{s}\pi}$, on the other hand we have $U(0,s) = \int_0^{+\infty} e^{-st} u(0,t)dt = 0$,
i.e.,

$$c_1 + c_2 = 0. \tag{3.4}$$

And $U(\pi,s) = \int_0^{+\infty} e^{-st} u(\pi,t)dt = 0$
i.e.,

$$c_1 e^{\sqrt{s}\pi} + c_2 e^{-\sqrt{s}\pi} = 0. \tag{3.5}$$

From equation (3.4) and equation (3.5), we obtain

$$c_1 = c_2 = 0.$$

Then the general solution is

$$U(x,s) = \frac{\sin(x)}{1+s}.$$

Applying the inverse Laplace transform, then we have

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}(U(x,s))(t) \\ &= \mathcal{L}^{-1}\left(\frac{\sin(x)}{1+s}\right)(t) \\ &= e^{-t}\sin(x). \end{aligned}$$

Exercise 3.10. Given the Gamma function

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad x > 0.$$

1) Show that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n+1) = n!$, $n \in \mathbb{N}$.
2) Calculate the Laplace transform of the function $x^\alpha H(x)$, where $H(x)$ is the Heaviside function and $\alpha > -1$.
3) Deduce $\mathcal{L}(x^n H(x))(s)$.
4) Calculate $\mathcal{L}(\sqrt{x})(s)$.

Solution 1) We have

$$\begin{aligned}
\Gamma(x+1) &= \int_0^{+\infty} t^x e^{-t}dt \\
&= [-t^x e^{-t}]_0^{+\infty} + x\int_0^{+\infty} t^{x-1}e^{-t}dt, \text{ by part} \\
&= x\Gamma(x).
\end{aligned}$$

And

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \\
&= n(n-1)\Gamma(n-1) \\
&= n(n-1) \times ... \times 2 \times 1 \times \Gamma(1),
\end{aligned}$$

and $\Gamma(1) = \int_0^{+\infty} e^{-t}dt = 1$, which implies that $\Gamma(n+1) = n!$.
2) We have

$$\begin{aligned}
\mathcal{L}(x^\alpha H(x))(s) &= \int_0^{+\infty} e^{-sx} \cdot x^\alpha dx, \text{ we put } sx = t \Leftrightarrow dt = sdx \\
&= \int_0^{+\infty} e^{-t}(\frac{t}{s})^\alpha \frac{dt}{s} \\
&= \frac{1}{s^{\alpha+1}} \cdot \Gamma(\alpha+1).
\end{aligned}$$

3) If $\alpha = n$, we have

$$\begin{aligned}
\mathcal{L}(x^n H(x))(s) &= \frac{\Gamma(n+1)}{s^{n+1}} \\
&= \frac{n!}{s^{n+1}}
\end{aligned}$$

4) If $\alpha = \frac{1}{2}$, we have

$$\begin{aligned}\mathcal{L}(\sqrt{x})(s) &= \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{3}{2}}}\\ &= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{\frac{3}{2}}}.\end{aligned}$$

we have

$$\begin{aligned}\Gamma(\frac{1}{2}) &= \int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt, \text{ we put } t = x^2 \Leftrightarrow dt = 2xdx\\ &= 2\int_0^{+\infty} e^{-x^2}dx\\ &= 2\cdot\frac{\sqrt{\pi}}{2}\\ &= \sqrt{\pi}.\end{aligned}$$

So

$$\mathcal{L}(\sqrt{x})(s) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.$$

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